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ELEMENTS OF ALGEBRA

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PREFACE

The author has aimed to make this treatment of algebra so simple that the pupil can begin the book to advantage immediately upon completing an ordinary course in arithmetic; and, at the same time, so scientific that he will have nothing to unlearn as he advances in the study of mathematics. Great care has been taken to develop the subject logically, yet the immaturity of the pupil has been constantly kept in mind, and every legitimate aid has been given him. Simplicity has been attained not by using inexact statements and mechanical methods, but by avoiding many of the outgrown phrases of traditional algebra, by giving demonstrations and explanations in full, and by making fundamental concepts clear and tangible. An introductory chapter explains the meaning and advantages of the literal notation, and illustrates the use of the equation in solving arithmetic problems. In Chapter II real numbers are first considered, and are defined as multiples of the quality-units, $+1$ and -1, and the pupil is drilled in the use of particular real numbers before he is required to represent general real numbers by letters.

General principles are first illustrated by *particular* examples, the study of which prepares the pupil to grasp the meaning of the formal statement of the principles, and

makes it less likely that he will memorize without comprehending the demonstrations which follow. With this arrangement the reproduction of the demonstrations may be left for the review; but the pupil should become familiar with each principle and definition before a new one is considered. When the demonstrations are not reproduced, it is recommended that the proofs be carefully read and discussed in class, so that the pupil may be fully convinced that the principles are true. He should then be required to state the authorities for each step in the proof when the steps are given.

The identity and the equation are sharply distinguished. Two groups of principles are stated, the first for *proving* the identity, the second for *solving* the equation.

The need of the principles of the equivalency of equations and systems is clearly shown. These principles are fully illustrated and proved, and upon them are based the methods of solving equations and systems of equations. In the chapter on factoring, the formation of equations with given roots serves as an introduction to the converse problem of finding the roots of a given quadratic or higher equation, and to the method of making factoring fundamental in the study and solution of quadratic and higher equations and systems.

The graph is used to illustrate the meaning of equations in two unknowns, of systems of equations and of equivalent systems; it also serves to make clear some of the general properties of equations in one unknown.

The theory of limits is given as briefly as is thought to be consistent with clearness. It is used in proving the

laws of incommensurable numbers and in evaluating expressions which assume the indeterminate form 0/0.

The treatment of imaginary numbers affords a good illustration of the advantages derived from regarding algebraic numbers as arithmetic multiples of quality-units. When a pupil understands that the quality-units $\sqrt{-1}$ and $-\sqrt{-1}$ include the idea of the arithmetic one and that of oppositeness to each other, that $(\sqrt{-1})^2 \equiv -1$, and $(\sqrt{-1})^4 \equiv +1$, he has mastered all that is new in imaginaries, and can then state the general laws for products and quotients of imaginary and real numbers (§§ 274, 276). This concept makes for simplicity, for it enables us to express general laws which are true for real, imaginary, and complex numbers, and it clearly separates the problem of finding the arithmetical value of a result from that of finding its quality. Graphic representations are used to illustrate the meaning and reality of imaginary and complex numbers.

Special attention is invited to the brevity and completeness of the demonstrations of the principles of proportion, the early introduction of the remainder theorem, the use of type-forms in factoring, and the treatment of fractional and irrational equations.

The methods of working examples have been chosen for their simplicity and the scope of their application. The problems are varied, interesting, well graded, and not so difficult as to discourage the beginner. Many exercises contain easy examples which, especially in the review, should be used for oral work. Suggestions as to the method of attack are freely given; rules are stated only

for the most difficult operations, but not until after these have been illustrated by particular examples.

The author has sought to treat each subject with sufficient fulness to meet the college entrance requirements, and more subjects are given than are ordinarily considered as a part of elementary algebra.

The author is indebted to many teachers for valuable suggestions, but especially to his assistant, Mr. C. D. Kingsley, who has carefully read all the manuscript and most of the proof sheets.

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COLGATE UNIVERSITY,
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CONTENTS

CHAPTER I

Introduction

CHAPTER II

Positive and Negative Numbers

CHAPTER III

Addition, Subtraction, and Multiplication of Real Numbers

CHAPTER IV

Addition and Subtraction of Integral Literal Expressions

CHAPTER V

Multiplication of Integral Literal Expressions

CHAPTER VI

Division of Integral Literal Expressions

CHAPTER VII

Linear Equations in One Unknown

CHAPTER VIII

Problems solved by Linear Equations in One Unknown

CHAPTER IX

Powers, Products, Quotients

CHAPTER X

Factors of Integral Literal Expressions

CHAPTER XI

Highest Common Factor and Lowest Common Multiple

CHAPTER XII

Fractions

CHAPTER XIII

Fractional Equations

CHAPTER XIV

Systems of Linear Equations

CHAPTER XV

Problems solved by Systems

CHAPTER XVI

Evolution. Irrational Numbers

CHAPTER XVII

Surds

CHAPTER XVIII

Imaginary and Complex Numbers

CHAPTER XIX

Quadratic Equations in One Unknown

CHAPTER XX

Problems

CHAPTER XXI

Irrational Equations

CHAPTER XXII

Higher Equations

CHAPTER XXIII

Systems involving Quadratic and Higher Equations

CHAPTER XXIV

Inequalities

CHAPTER XXV

Ratio and Proportion

CHAPTER XXVI

Theory of Exponents

CHAPTER XXVII

Indeterminate Equations and Systems

CHAPTER XXVIII

Theory of Limits

CHAPTER XXIX

The Progressions

CHAPTER XXX

Permutations and Combinations

CHAPTER XXXI

Binomial Theorem

CHAPTER XXXII

Logarithms

CHAPTER XXXIII

Graphic Solution of Equations and Systems

CHAPTER XXXIV

Theory of Equations

ELEMENTS OF ALGEBRA

CHAPTER I

INTRODUCTION

1. Arithmetic number. In Arithmetic we have seen that by taking a group of *ones* we can obtain any *whole* number; and that by dividing *one* into equal parts, and taking a group of these parts, we can obtain any *fractional* number.

Hence, the **primary unit** of arithmetic number is **one**, 1.

A **whole number**, or an **integer**, is one or an aggregate of ones.

A **fractional unit** is one of the equal parts of one.

A **fractional number** is a fractional unit, or any aggregate of fractional units which does not equal a whole number.

In writing fractions we often use the sign /; thus, 3/2 denotes $\frac{3}{2}$.

The numbers defined above answer the single question, 'How many?' and are called **arithmetic**, or *absolute*, **numbers**.

Arithmetic numbers are used to express how many times one quantity contains another of the same kind. By diminishing indefinitely the fractional unit we can obtain a series of numbers in which the difference between successive numbers will be as small as we please.

2. A **numeral** is any symbol which is used to denote a particular number, and which is never used to denote any other number. The more common numerals are the Arabic figures, 1, 2, 3, etc., and the Roman letters, I, V, X, etc.

By the use of numerals, as we have seen in Arithmetic, we can state only *particular* problems. To state and solve *general* problems, and to investigate the general properties of numbers, mathematicians have invented the literal notation.

3. Letters denoting numbers. — An important step in enlarging the notation of number is the use of a letter, as a, b, x, or y, to denote *any number whatever* or *an unknown number.*

E.g., just as heretofore we have spoken of 5 dollars, of $8\frac{1}{2}$ miles, etc., so sometimes we shall speak of a dollars, meaning *any number whatever* of dollars; of x miles, meaning *any number* of miles or *an unknown number* of miles, etc.

Just as when we say the number 4, or simply 4, we mean *the number denoted by the figure* 4; so when for brevity we say the number a, or simply a, we mean *the number denoted by the letter* a.

The following simple examples will illustrate how *letters* are used to denote *any number whatever* in the statement of general arithmetic problems.

Ex. 1. If one merchant has 50 dollars and another has 25 dollars, the two together have $50 + 25$ dollars. If one merchant has m dollars and another has n dollars, the two together have $m + n$ dollars.

Here m or n denotes any whole or fractional number; and $m + n$ denotes the sum of these numbers.

Ex. 2. If a drover buys 5 horses at 50 dollars each, he pays 50×5 dollars for the horses. If a drover buys y horses at x dollars apiece, he pays $x \times y$ dollars for the horses.

Here y denotes any whole number, x any whole or fractional number, and $x \times y$ their product.

Ex. 3. If 60 dollars is divided equally among 5 boys, each boy receives $60 \div 5$ dollars. If x dollars is divided equally among n boys, each boy receives $x \div n$ dollars.

Ex. 4. If m men earn n dollars in one day, each man earns $n \div m$ dollars in one day, and therefore x men will earn $n \div m \times x$ dollars in one day.

In these examples the reasoning is the same whether the numbers are denoted by figures or by letters.

When letters are used in its statement, each problem is a *general* problem, and includes an unlimited number of *particular* problems.

4. The following **signs**, or *symbols*, **of operation**, with which, as has been assumed, the pupil is already familiar, are common to all branches of mathematics.

The **sign of addition**, $+$, read '*plus*,' indicates that the number after the sign is to be added to the number before it.

E.g., $3 + 4$ means that 4 is to be added to 3. $a + b$, read 'a plus b,' means that the number denoted by b is to be added to the number denoted by a; or, more briefly, it means that b is to be added to a.

The **sign of subtraction**, $-$, read '*minus*,' indicates that the number after the sign is to be *subtracted* from the number before it.

E.g., $a + b - c$, read 'a plus b minus c,' means that b is to be added to a, and then c subtracted from this sum.

The **sign of multiplication**, $\times$, or a point above the line, read '*multiplied by*,' or '*into*,' indicates that the number before it is to be multiplied by the number after it.

The sign of multiplication is usually omitted between two letters or a figure and a letter.

E.g., $2\,ab$, read '$2\,ab$,' means $2 \times a \times b$; $7\,abc$, read '$7\,abc$,' means $7 \cdot a \cdot b \cdot c$. The sign of multiplication cannot be omitted between two factors when *both* are denoted by figures; for by the notation of Arithmetic, 54 means $50 + 4$, not 5×4.

The **sign of division**, $\div$, read '*divided by*,' or '*by*,' indicates that the number before it is to be divided by the number after it.

E.g., $a \div b \times c \div d$, read 'a by b into c by d,' denotes that a is to be divided by b, the result multiplied by c, and then this result divided by d.

Observe that in a series of *additions and subtractions*, or in a series of *multiplications and divisions, the operations are to be performed from left to right.*

Exercise 1.

1. If a boy has 5 marbles and wins 4 more, how many marbles has he? If he has a marbles and wins b more, how many marbles has he?

2. One part of 25 is 7. What is the other part? One part of 25 is n. What is the other part? One part of the number m is n. What is the other part?

3. The difference of two numbers is 6, and the smaller is 12. What is the greater? The difference of two numbers is n, and the smaller is x. What is the greater?

4. How old will a man be in 6 years, if his present age is 36 years? How old will a man be in c years, if his present age is x years?

5. In 10 years a man will be 50 years old. What is his present age? In b years a man will be m years old. What is his present age?

6. The length of a room is x feet, and its width is b feet less than its length. What is its width?

7. One number is x, and a second number is y times as great. What is the second number?

8. One number, x, is y times as great as a second number. What is the second number?

9. The number which contains 4 units and 5 tens is $10 \times 5 + 4$. Write the number which contains x units and y tens.

10. Write a number containing x units, y tens, and v hundreds.

11. Of three consecutive whole numbers 6 is the second; what are the first and the third? If the second is m, what are the first and the third?

12. Of three consecutive whole numbers 7 is the first; what are the second and the third? If the first whole number is x, what are the second and the third?

13. Of three consecutive even integers, 8 is the third; what are the first and the second? If the third integer is m, what are the first and the second?

14. If a goat costs x dollars, and a cow costs 4 times as much as a goat, and a horse costs 3 times as much as a cow, how much does a horse cost?

15. In example 14, how much do a goat, a cow, and a horse together cost?

16. A is x years old, B is 17 years older than A, and C's age equals the sum of B's age and A's age. How old is C?

17. If m sheep cost x dollars, and n cows cost y dollars, what would c sheep and b cows cost?

18. A travelled a hours at the rate of m miles an hour, and B travelled b hours at the rate of y miles an hour. How many miles did A and B together travel?

19. A rides his bicycle n yards; the circumference of each wheel is m feet. How many revolutions does each wheel make in going this distance?

5. A **mathematical expression** is any symbol or combination of symbols which denotes a number.

If all the symbols of number in an expression are numerals, the expression is called a **numeral expression.**

An expression which involves one or more letters is called a **literal expression.**

The number denoted by a *numeral expression* is a *particular*, or a *fixed*, number. For sake of distinction, the

number which is denoted by a *literal* expression is called a *general*, or an *arbitrary*, number.

By the **value** of an expression we mean the number denoted by it.

E.g., 4, $5-3$, and $7 \times 5 + 4 \times 2$ are *numeral* expressions, and each denotes a *particular*, or fixed, number; while a, $a+4$, and $ax+b-c \div y$ are *literal* expressions, and each denotes a *general* number.

6. An **axiom** is a truth so obvious that it may be taken for granted.

Two numbers are said to be **equal** when they bear the same relation to the same unit.

E.g., 4×3 and 6×2 are equal numbers, since each is 12 times 1. $\frac{2}{3}$ and $\frac{6}{9}$ are equal numbers, since each is 6 times $\frac{1}{9}$.

The axioms concerning equal numbers, which are most frequently used in Algebra, as in Arithmetic, are the following:

1. *Any number is equal to itself.*

2. *Any number is equal to the sum of all its parts.*

3. *If each of two numbers is equal to the same number, they are equal to each other.*

4. *If equal numbers are added to equal numbers, the sums are equal.*

5. *If equal numbers are subtracted from equal numbers, the remainders are equal.*

6. *If equal numbers are multiplied by equal numbers, the products are equal.*

7. *If equal numbers are divided by equal numbers*, **except zero**, *the quotients are equal.*

E.g., $12 = 8 + 4$, and $12 \div 4 = (8 + 4) \div 4$.
Again, $2 \times 0 = 5 \times 0$; but we cannot divide by 0 and say that $2 = 5$.

8. *The value of a mathematical expression is not changed when, for any number in it, an equal number is substituted.*

7. The following **signs of relation** are common to all branches of mathematics:

The **sign of equality,** $=$, read *'is equal to,'* is placed between two expressions to indicate that they denote equal numbers.

The **sign of inequality,** $>$, read *'is greater than,'* is placed between two expressions to indicate that the first denotes a *greater* number than the second. The sign $<$ is read *'is less than.'*

E.g., $4+8>10$ is read '4 plus 8 *is greater than* 10';

and $7-2<12$ is read '7 minus 2 *is less than* 12.'

Observe that in each case the small end of the symbol is toward the *less* number.

The sign $\neq$, read *'is not equal to,'* is used in stating that two numbers are unequal, without indicating which is the greater. Thus, $a \neq b$ is read 'a is not equal to b.'

8. The **signs of grouping** are the *parentheses* ( ), the *brackets* [ ], the *braces* { }, and the *vinculum* ——.

Each of these symbols indicates that the expression included by it is *to be treated as a whole.*

E.g., the expression $12-(3+5)$ denotes that the sum $3+5$ is to be subtracted from 12; that is,

$$12-(3+5)=12-8=4.$$

The expression $[32-(4+6)\div 5]\div 3$ denotes that one-fifth of the sum $4+6$ is to be subtracted from 32, and the remainder divided by 3; that is,

$$[32-(4+6)\div 5]\div 3=[32-2]\div 3=10.$$

When one sign of grouping is used within another, to avoid ambiguity different forms must be used as above.

9. Classification of expressions. A **term** is any expression in which the symbols of number are not connected by the sign $+$ or $-$; as $4\times 5\div 2$ or $3ab\div c$.

Hence the signs $\times$ and $\div$ indicate operations within a term, and the parts of an expression which are connected by the sign $+$ or $-$ are its terms.

E.g., each of the expressions 5, a, and $5x \div a$ is a term.

The expression $2ax + 3b \div c$ consists of two terms, $2ax$ and $3b \div c$.

In this definition of a term an expression *within a sign of grouping* must be considered as a single symbol of number. Hence a factor or a divisor in a term can itself consist of two or more terms.

E.g., the expression $(a+b)(c+d)$ is a term in which each of the factors, $a+b$ and $c+d$, consists of two terms.

A **monomial** is an expression of one term; as 4, $6xy$, or $7(a+b) \div (x+y)$.

A **polynomial** is an expression of two or more terms; as $4+7$ or $a+3xy+7b$.

A polynomial of two terms is called a **binomial.**

A polynomial of three terms is called a **trinomial.**

Observe that all operations within each of two terms must be performed before performing the operation between them.

E.g., the binomial $10-(4+2)(7-3)\div(6+2)$ denotes that $4+2$ is to be multiplied by $7-3$, this product divided by $6+2$, and the resulting quotient subtracted from 10.

Exercise 2.

Express in its simplest form the number denoted by each of the following numeral expressions:

1. $14+(7-4)$.
2. $18-(12-7)$.
3. $(6+2)-(7-3)$.
4. $(3+8)3$.
5. $18-(6-2)3$.
6. $(6+9)\div 5$.
7. $16-(7-1)\div 3$.
8. $22-(18-6)\div 4$.

9. $12+[4-(5-3)]$.

10. $18-[8-(4+2)]$.

11. $19-[(2+4)-(5-3)]$.

12. $22-[23-(7-4)]\div 5$.

13. How many terms in each of the expressions found in examples 1 to 12 inclusive?

14. Find the sum of $5x$ and $7x$.

Just as $5+7=12$, so $5x+7x=12x$.

Reduce each of the following expressions to its simplest form:

15. $2x+4x$.

16. $5x+7x$.

17. $8x+4x-6x$.

18. $9x-3x+2x$.

19. $\frac{1}{2}x+\frac{1}{4}x+\frac{5}{6}x$.

20. $\frac{2}{3}a+\frac{1}{2}a-\frac{1}{6}a$.

10. An **equality** is the statement that two expressions denote the same number. The expression to the left of the sign of equality is called the *first member* of the equality, and the expression to the right of this sign is called the *second member*.

E.g., $(5+3)9=72$ is an equality; of which $(5+3)9$ is the *first* member and 72 is the *second* member.

11. Zero is the number obtained by subtracting any number from itself; that is, zero is defined by the equality

$$a-a=0. \tag{1}$$

12. To find the value of a given literal expression when each of its letters has some particular value, we substitute for each letter its particular value, and simplify the resulting expression.

Ex. Find the value of the expression $(x+y)z\div(a-b)$, when $x=6$, $y=3$, $z=4$, $a=9$, $b=2$.

Substituting, 6 for x, 3 for y, 4 for z, 9 for a, and 2 for b, in the given expression, we obtain

$$(x+y)z\div(a-b)=(6+3)\times 4\div(9-2) \tag{1}$$

$$=36\div 7. \tag{2}$$

In the work above we have three equalities; by axiom 8, the first expression is equal to the second and the second is equal to the third; hence, by axiom 3, the first is equal to the third.

13. In working examples the student should give heed to the following suggestions:

1. Too much importance cannot be attached to neatness of style and arrangement. Neatness is in itself conducive to accuracy.

2. It should be clearly brought out how each result follows from the one before it; for this purpose it will sometimes be advisable to add short verbal explanations.

3. Unless the members are very short the signs of equality in the steps of the work should be placed one under the other.

Exercise 3.

Find the value of each of the following expressions when $a = 5,\ b = 3,\ c = 4,\ x = 6$:

1. $a + b$.
2. $a - b$.
3. $a + b - c$.
4. abc.
5. $ab \div c$.
6. $(a + b)x$.
7. $(a - b)c$.
8. $(a + b) \div x$.
9. $(a - b) \div x$.
10. $x \div (a + c)$.
11. $x \div (a - c)$.
12. $(a + b)(c + x)$.
13. $(a - b)(x - c)$.
14. $[x - (b + 1)]a$.
15. $[x + (a - c)] \div a$.
16. $[3b - (x - a)] \div c$.
17. $(3a - 2b) \div (x - b)$.
18. $(9 - a)(2b - c)(2x - 3b)$.
19. $(3x - 4c)(3b - 2c) \div (x - c)$.
20. $[2a - (3b - 2c)] \div [(3c - 3b)(2a - 3b)]$.
21. $[3x - 2(a - b)] \div [(2x - 3b)(a - c)]$.
22. $(2x - 3b)(4a - 3x) \div (3x - 3c - b)$.

14. A **proof** is a course of reasoning by which the truth of a statement is made clear, or is established.

15. Identical expressions. Two *numeral* expressions which denote the same number or *any* two expressions which denote equal numbers for all values of their letters are called *identical expressions.*

E.g., the *numeral* expressions $36 \div 4$ and $13 - 4$ are identical, for each denotes the number 9.

Again, the *literal* expressions $3x + 7x$ and $6x + 4x$ are identical, for each denotes the general number $10x$.

To *prove that two expressions are identical*, we reduce one to the form of the other, or we reduce both to the same form.

Ex. Prove that the expressions $7x + 3x + 2x$ and $14x - 2x$ are identical.

$7x + 3x + 2x$ denotes $12x$, and $14x - 2x$ denotes $12x$; hence, by definition, the two expressions are identical.

An equality whose members are identical expressions is called an **identity.**

The **sign of identity,** $\equiv$, read *'is identical with,'* is often used instead of the sign $=$ in writing a *literal* identity, *i.e.*, one whose members involve one or more letters.

E.g., $$9 + 6 = 5 \times 3, \tag{1}$$
or $$3x + 7x \equiv 8x + 2x, \tag{2}$$
is an identity, (1) being *numeral* and (2) being *literal.*

Any equality which involves only numerals is an identity.

The sign $\equiv$ points out the fact that equality (2) is an identity.

The pupil should now prove the identities in Exercise 4.

16. Letters denoting unknowns. Any problem involves one or more numbers whose values are *given*, and one or more numbers whose values are *to be found.* Numbers *given* are called *knowns*, numbers *to be found* are called *unknowns.* An unknown is usually denoted by one of the last letters of the alphabet; as x, y, z.

The following simple problems illustrate the advantage of denoting an *unknown* by a letter.

Prob. 1. The sum of two numbers is 80, and the greater is 3 times the less. Find the numbers.

Let $x =$ the less number;

then, since the greater is three times the less,

$3x =$ the greater number.

Hence their sum $= x + 3x = 4x$.

Therefore, by the *conditions* of the problem, we have

$$4x = 80. \quad (1)$$

Divide by 4, $x = 20$, the less number.

Multiply by 3, $3x = 60$, the greater number.

Observe that the numbers 20 and 60 satisfy the *conditions* of the problem; that is, $20 + 60 = 80$, and $60 = 20 \times 3$.

Prob. 2. A farmer bought a horse, a cow, and a goat; the horse cost 3 times as much as the cow, and the cow 4 times as much as the goat, and all three together cost 255 dollars. What was the cost of each?

Let $x =$ the *number* of dollars the goat cost;

then $4x =$ the *number* of dollars the cow cost,

and $12x =$ the *number* of dollars the horse cost.

Hence the *number* of dollars all three cost

$$= x + 4x + 12x = 17x.$$

Therefore, by the *conditions* of the problem, we have

$$17x = 255. \quad (2)$$

Divide by 17, $x = 15$.

Multiply by 4, $4x = 60$.

Multiply by 3, $12x = 180$.

Hence the goat cost \$15, the cow \$60, and the horse \$180.

17. Equations. Any equality which is not an identity is called an *equation*, as (1) or (2) in § 16.

A *value of* x in an equation in x is any number which when substituted for x makes the equation an identity.

An equation in one unknown as x restricts x to one value or to a definite number of values.

E.g., if in the equation

$$2x + 2 = x + 8, \tag{1}$$

we put 6 for x, we obtain the identity

$$2 \times 6 + 2 = 6 + 8.$$

Hence 6 is a *value* of x in equation (1); and, as will be seen later, 6 is the *only* value of x in (1).

An equation, as (1) or (2) in § 16, *expresses* in symbols *the conditions* of a problem; and it *restricts* its unknown to such values as will satisfy these conditions.

Thus the equation $4x = 80$ restricts x to the one value 20; and the equation $3x = 15$ restricts x to the one value 5.

The two kinds of equalities, *equations* and *identities*, must be clearly distinguished the one from the other.

An equation states a condition, and the values of the unknown which satisfy this condition are to be found; while an identity states that one of two expressions can be reduced to the other, and is to be proved.

Exercise 4.

Prove each of the following identities:

1. $7 \times 3 \times 2 = (10 - 3) \times 6$.
2. $88 \div 4 = (7 + 4) \times 2$.
3. $204 \div 6 = (12 + 5) \times 2$.
4. $2x + 7x \equiv (27 \div 3) \cdot x$.
5. $9a + 15a \equiv (6 \times 4) \cdot a$.
6. $10b + 8b \equiv (36 \div 2) \cdot b$.

By *inspection* find a value of x in each of the following equations, and verify it by substitution:

7. $x - 4 = 0$.
8. $2x = 14$.
9. $3x - 15 = 0$.
10. $4x = 20$.
11. $2x + 1 = 7$.
12. $2x + 4 = 8$.
13. $3x + 1 = 10$.
14. $x - 1 = 6$.
15. $2x - 4 = 4$.
16. $2x + 1 = x + 3$.
17. $2x - 1 = x + 2$.

18. The following principles, which are proved in Chapter VII, are used in finding the values of the unknown in an equation:

(i) *If the same number is added to or subtracted from both members of an equation, the unknown has the same values in the derived equation as in the given one.*

(ii) *If both members of an equation are multiplied or divided by the same known number* (**except zero**), *the unknown has the same values in the derived equation as in the given one.*

Ex. 1. Find the value of x in the equation

$$2x + 5 = 11. \qquad (1)$$

Subtracting 5 from each member, we remove all the known terms from the first member, and obtain

$$2x = 6. \qquad (2)$$

Dividing each member by 2, we obtain

$$x = 3. \qquad (3)$$

By principle (i), x has the same value in (2) as in (1); and by (ii), x has the same value in (3) as in (2).

Hence 3 is the one and only value of x in equation (1).

Ex. 2. Find the value of x in the equation

$$4x - 2 = x + 4. \qquad (1)$$

Add 2, $$4x = x + 6. \qquad (2)$$

Subtract x, $$3x = 6. \qquad (3)$$

Divide by 3, $$x = 2. \qquad (4)$$

By principle (i), x has the same value in (2) as in (1), and the same in (3) as in (2); by (ii), x has the same value in (4) as in (3). Hence 2 is the one and only value of x in (1).

Check: Putting 2 for x in (1), we obtain the identity

$$4 \times 2 - 2 = 2 + 4.$$

Hence 2 is a value of x in (1).

Ex. 3. Find the value of x in the equation

$$\tfrac{3}{2}x - \tfrac{5}{4}x = \tfrac{7}{8}. \qquad (1)$$

To clear (1) of fractions, we multiply both its members by 8, *i.e.*, by the least common multiple of its denominators.

$$12x - 10x = 7, \text{ or } 2x = 7. \qquad (2)$$

Divide by 2, $$x = \tfrac{7}{2}. \qquad (3)$$

By (ii), x has the same values in (2) as in (1), and the same in (3) as in (2); hence $\frac{7}{2}$ is the one and only value of x in (1).

The foregoing examples illustrate the method of finding the value of the unknown in a simple equation.

Exercise 5.

Find the value of x in each of the following equations:

1. $3x - 7 = 2x + 3.$
2. $3x + 4 = x + 10.$
3. $4x + 4 = x + 7.$
4. $7x + 5 = x + 23.$
5. $8x = 5x + 42.$
6. $6x - 5 = 4x + 1.$
7. $18x - 7 = 43 - 7x.$
8. $5x - 7 = 3x + 1.$
9. $19x - 11 = 15 + 6x.$
10. $3x + 15 = x + 25.$
11. $5x - 2 = 3x + 4.$
12. $7x - 9 = 17 + 2x.$
13. $\frac{2}{3}x - 4 = 5 - \frac{1}{3}x.$
14. $\frac{4}{5}x - 3 = 7 - \frac{1}{5}x.$
15. $\frac{7}{8}x - \frac{1}{2} = \frac{1}{3} - \frac{1}{8}x.$
16. $\frac{3}{4}x - \frac{1}{5} = \frac{5}{6} - \frac{1}{4}x.$
17. $7x + 21 = 45 - 5x.$
18. $x + \frac{3}{4} = \frac{41}{12} - \frac{1}{2}x.$
19. $\frac{1}{2}x + 1 = \frac{1}{3}x + \frac{9}{4}.$
20. $\frac{6}{13}x + \frac{12}{13} = \frac{4}{13}x + 4.$

19. Problems solved by equations. Read the problem carefully to find out exactly what it means; then state in algebraic symbols just what it says.

To do this, let x denote the unknown number; or, if there are two or more unknown numbers, let x or some multiple of x denote one of them, and then express each of the others in terms of x.

By an equation express the condition which the problem imposes on x.

Then find the value of x in this equation.

Exercise 6.

1. A line 30 inches long is divided into two parts, one of which is double the other. How long are the parts?

Let $x =$ the *number* of inches in the second part;

then $2x =$ the *number* of inches in the first part.

Hence the *number* of inches in the two parts $= 2x + x = 3x$.

Therefore, by the conditions of the problem, we have

$$3x = 30.$$

Divide by 3, $x = 10$, *number* in second part.

Multiply by 2, $2x = 20$, *number* in first part.

2. A, B, and C together have \$90. B has twice as much as A, and C has as much as A and B together. How much has each?

Let $x =$ the *number* of dollars A has;

then $2x =$ the *number* of dollars B has;

hence $3x =$ the *number* of dollars C has.

$$\therefore x + 2x + 3x = 90.$$

3. The sum of the ages of A and B is 67 years, and A is 17 years older than B. What is the age of each?

Ans. 42 and 25 years.

4. Three men, A, B, and C, trade in company and gain \$600, of which A is to have 3 times as much as B, and C as much as A and B together. What is the share of each?

Let $x =$ the number of dollars B is to have, etc.

5. A farmer bought 3 cows for \$180, and the prices paid were as the numbers 1, 2, and 3. What was the cost of each?

Let $\qquad x =$ the *number* of dollars paid for the first;

then $\qquad 2x =$ the *number* of dollars paid for the second,

and $\qquad 3x =$ the *number* of dollars paid for the third.

6. Divide 500 into two parts which are as the numbers 1 and 4.

7. What number is that whose double exceeds its half by 27?

8. Divide \$ 575 between A and B so that A may receive \$ 75 more than B.

Let $\qquad x =$ the *number* of dollars B receives;

then $\qquad x + 75 =$ the *number* of dollars A receives;

hence $\qquad 2x + 75 = 575. \qquad (1)$

9. Divide 105 into two parts whose difference is 45.

10. What number is that to which if 40 is added the sum will be 3 times the original number?

11. Divide \$ 84 among A, B, and C, so that B shall have \$ 13 more than A, and C \$ 16 more than B.

12. Three men, A, B, and C, contribute to an enterprise \$ 2400. B put in twice as much as A, and C put in as much as A and B together. How much did each contribute?

13. Find two numbers whose difference is 10, and one of which is 3 times the other.

14. If two men, 150 miles apart, travel toward each other, one at the rate of 2 miles an hour, and the other at the rate of 3 miles an hour, in how many hours will they meet?

15. A horse, carriage, and harness together are worth \$ 625. The horse is worth 8 times as much as the harness, and the carriage is worth \$ 125 more than the harness. Find the value of each. *Ans.* \$ 400, \$ 175, and \$ 50.

16. A man bought a cow, a sheep, and a hog for \$ 80; the cow cost \$ 32 more than the sheep, and the sheep \$ 6 more than the hog. Find the price of each.

Ans. \$ 50, \$ 18, \$ 12.

17. The sum of \$ 6000 was divided among A, B, C, and D; B received twice as much as A, C as much as A and B together, and D as much as A, B, and C together. How much did each receive?

Ans. \$ 500, \$ 1000, \$ 1500, \$ 3000.

18. A man has two sons and one daughter. He wishes to divide \$ 12,000 among them so that the younger son shall have twice as much as the daughter, and the older son as much as both the other children. How much must he give to each?

19. Divide 90 into five parts so that the second shall be 5 times the first, the third shall be $\frac{2}{3}$ of the first and second, the fourth shall be $\frac{1}{2}$ of the first, second, and third, and the fifth shall be 2 times the sum of the other four.

20. A, B, and C enter into partnership to do business. A furnishes 5 times as much capital as B, and C furnishes $\frac{1}{2}$ as much as A and B together. They all together furnish \$ 18,900. How much does each furnish?

21. A gentleman, dying, bequeathed his property of \$ 21,840 as follows: to his son 2 times as much as to his daughter, and to his widow $1\frac{1}{3}$ times as much as to both his son and daughter. What was the share of each?

22. A farmer purchased 100 bushels of grain. He bought 2 times as many bushels of corn as of oats, and $2\frac{1}{3}$ times as many bushels of wheat as of oats and corn. How many bushels of each kind did he buy?

23. Three candidates for an office polled the following votes respectively: B received 3 times as many votes as A, and C $1\frac{1}{2}$ times as many as A and B together. The whole

number of votes was 11,000. How many votes did each receive?

24. A banker loaned to each of 4 men equal sums of money. One man had the money 2 years, another $2\frac{1}{2}$ years, another $3\frac{1}{2}$ years, and another $4\frac{1}{2}$ years. The entire interest money received was $275. How much did each man pay?

Let $x =$ the number of dollars in the yearly interest on the sum loaned to each man.

25. A library contains 9 times as many historical works, and 5 times as many scientific books, as works of fiction. The historical works exceed the works of fiction and science by 10,500 volumes. How many volumes are there of each?

26. A drover, being asked how many sheep he had, replied that if he had 3 times as many as he then had and 6 more, he would have 150. How many had he?

27. The expenses of a manufacturer for 5 years were $17,500. If they increased $500 annually, what were his expenses each of the five years?

28. A farmer had 590 sheep distributed in three fields. In the first field there were 25 more than in the second, and in the third there were 15 more than in the first. How many sheep were in each field?

29. Of a herd of cows, 280 are Jerseys, and these are 35% of the entire herd. How many cows in the herd?

Let $x =$ the *number* of cows in the entire herd; then $\frac{35}{100}x = 280$.

30. A town lost 7% of its inhabitants, and then had 6045 inhabitants. What was its population before the loss?

31. What number increased by $\frac{1}{5}$ of 25% of itself equals 315?

32. The annual rent of a house is $240, and this is 8% of its value. What is its value?

CHAPTER II

POSITIVE AND NEGATIVE NUMBERS

20. **Algebra** treats of the *equation*, its nature, the methods of solving it, and some of its applications.

21. In each of the equations thus far considered, the unknown is an arithmetic number. But in many equations the unknown cannot be an arithmetic, or absolute, number.

E.g., take the equation

$$3x = 2x - 5. \tag{1}$$

Subtracting $2x$ from each member of (1), we obtain

$$x = 0 - 5, \text{ or } -5. \tag{2}$$

Hence the value of x in equation (1) is denoted by the expression -5, which has no meaning in Arithmetic.

If, therefore, such an equation as $3x = 2x - 5$ is to be of any use, we must so enlarge our concept of number as to give a meaning to such an expression as -5.

To gain this larger idea of number let us first consider *opposite* concrete quantities.

22. **Positive and negative, or opposite, quantities.** Two quantities are said to be *opposites*, if, when combined (or united as parts into one whole), any amount of the one destroys, or annuls, an equal amount of the other.

Of two opposite quantities, we call one *positive* and the other *negative*.

E.g., debts and credits are opposites; for when they are combined, any amount of debt annuls an equal amount of credit. If we call credits positive, debts will be negative.

Two forces acting in opposite directions are opposites; for when they are combined, any amount of the one annuls an equal amount of the other. If one of these forces is called positive, the other is called negative.

Distances measured or travelled in opposite directions are opposites; for when they are combined, any distance travelled in the one direction annuls an equal distance travelled in the opposite direction. If one distance is called positive, the other is negative.

The sign + or the sign − is often written before the measure of a concrete quantity to denote its **quality**, as *positive* or *negative*. When thus used, the signs + and − are read 'positive' and 'negative,' respectively, and are called *signs of quality*.

E.g., if we call credits positive, + \$5 will denote \$5 of credit, and − \$4 will denote \$4 of debt. If + 8 inches denotes 8 inches to the right, − 9 inches will denote 9 inches to the left. If + 3° denotes 3° above the zero point, − 7° will denote 7° below that point.

If + 400 years denotes 400 years after Christ, − 300 years will denote 300 years before Christ.

In this chapter and the next we shall use as *signs of quality* the small signs $^+$ and $^-$, which, by their size and position, are clearly distinguished from the *signs of operation*, + and −.

Exercise 7.

1. If credits are regarded as positive, what is denoted by $^+\$8$? By $^-\$11$? By $^+\$125$? By $^-\$175$?

If debts are regarded as positive, what does each of the above expressions denote?

2. If degrees above the zero point are regarded as positive, what is denoted by $^+1°$? By $^+22°$? By $^-5°$? By $^-20°$?

3. If distances measured from the point O to the right are regarded as positive, what is denoted by $^-7$ inches? By $^+14$ inches? By $^-13$ inches?

4. If distances north of the equator are regarded as positive, what is denoted by $^+300$ miles? By $^-700$ miles?

State in symbols each of the following in two ways:

5. \$45 gain and \$25 loss is equal to \$20 gain.

$$^+\$45 + {}^-\$25 = {}^+\$20, \text{ gain being positive;}$$

or, $$^-\$45 + {}^+\$25 = {}^-\$20, \text{ loss being positive.}$$

6. \$25 gain and \$30 loss is equal to \$5 loss.

23. Positive one and negative one. Just as from the concrete unit \$1 or 1° we gain the idea of the unit 1, so from the concrete positive and negative units $^+\$1$ and $^-\$1$, or $^+1°$ and $^-1°$, we gain the idea of *positive one*, $^+1$, and *negative one*, $^-1$.

Positive one, $^+1$, and *negative one*, $^-1$, include both the idea of the *arithmetic one* and that of *oppositeness* to each other.

The units $^+1$ and $^-1$ being opposites, each annuls the other when added to it; that is, $^+1 + {}^-1 = 0$, and $^-1 + {}^+1 = 0$.

The units $^+1$ and $^-1$ are called **quality-units.**

Of quality-units, $^+1$ is taken as the *primary* unit.

24. Positive and negative numbers. Just as we say that $^+4$ denotes 4 times $^+1$, or 4 positive units; so, enlarging the meaning of *times*, we shall say that $^+(\frac{2}{3})$ denotes $\frac{2}{3}$ times $^+1$, or $\frac{2}{3}$ a positive unit, and $^-(\frac{5}{2})$ denotes $\frac{5}{2}$ times $^-1$, or $\frac{5}{2}$ negative units.

Any arithmetic number of times the unit $^+1$ is called a *positive number*, as $^+5$. Any arithmetic number of times the unit $^-1$ is called a *negative number*, as $^-4$ or $^-(\frac{3}{2})$.

Observe that the only new idea in a positive or a negative number is that of the quality-unit $^+1$ or $^-1$.

A positive number and a negative number are *opposite numbers*. Thus $^+5$ and $^-4$ are opposite numbers.

A positive or a negative number answers the *two* questions, 'How many?' and 'Of what quality?' Its arith-

metic, or absolute, value answers the first question, and its quality-unit the second.

E.g., the *arithmetic* value of $^{+}5$ is 5, and its quality-unit is $^{+}1$; the arithmetic value of $^{-}(\frac{3}{5})$ is $\frac{3}{5}$, and its quality is negative.

A *positive* or a *negative number* is *integral* or *fractional* according as its arithmetic value is integral or fractional.

E.g., $^{+}(\frac{3}{2})$ and $^{-}(\frac{5}{8})$ are fractional numbers.

25. Symbols for positive and negative numbers. A figure (or figures) with the sign $^{+}$, or $^{-}$, prefixed denotes a *particular positive*, or a *particular negative* number. The figure denotes the arithmetic value, and the sign $^{+}$, or $^{-}$, denotes the quality-unit $^{+}1$, or $^{-}1$.

E.g., each of the expressions $^{+}3$, $^{-}7$, $^{+}8$, $^{-}5$ denotes a *particular* positive, or a *particular* negative, number.

A letter with the *small* sign $^{+}$ or $^{-}$ prefixed denotes a *general positive* or a *general negative* number. The letter denotes a general arithmetic number, and the sign $^{+}$ or $^{-}$, denotes the quality-unit $^{+}1$, or $^{-}1$.

E.g., the expression ^{+}a denotes a general positive number, the letter a denoting a general arithmetic number, and the small sign $^{+}$ the quality-unit $^{+}1$.

A letter not preceded by a *small* sign $^{+}$, or $^{-}$, denotes *any* number, positive or negative, integral or fractional.

E.g., a denotes $^{+}2$, $^{-}3$, $^{+}7$, $^{-}9$, or any other number, positive or negative; so also does b, x, y, or z.

Hence, a letter in Algebra denotes an *algebraic number* except when, by the presence of a *small* sign ($^{+}$ or $^{-}$) before it, it is restricted to an arithmetic value.

26. To **add** one number to another is to unite the one with the other into one whole or aggregate.

As in Arithmetic, the two given numbers are called **summands**, and the result is called the **sum.**

Ex. 1. Add $^+6$ to $^+4$.

Four times the unit $^+1$ plus 6 times the same unit is equal to $4+6$ times that unit; that is,

$$^+1 \times 4 + {}^+1 \times 6 = {}^+1\,(4+6);$$

or $$^+4 + {}^+6 = {}^+10.$$

Ex. 2. Add $^-5$ to $^-7$.

Seven times the unit $^-1$ plus 5 times the same unit equals $7+4$ times that unit; that is,

$$^-7 + {}^-5 = {}^-12.$$

These examples illustrate the following principle:

27. To add one number to another of the same quality, *find the sum of their arithmetic values and prefix to it the sign of their common quality.* Or stated in symbols,

$$^+a + {}^+b \equiv {}^+(a+b), \quad {}^-a + {}^-b \equiv {}^-(a+b).$$

Proof. a times the unit $^+1$, or $^-1$, plus b times the same unit is equal to $a+b$ times that unit.

Exercise 8.

1. What is the arithmetic (or absolute) value and the quality-unit of $^+7$? Of $^-15$? Of $^-11\frac{1}{2}$? Of ^-a? Of $^+(a+2)$? Of $^-(a+b)$?

2. Find the sum of $^+5$ and $^+7$. Of $^+3$ and $^+11$. Of $^-3$ and $^-16$. Of $^-7$ and $^-9$. Of $^-10$ and $^-12$. Of $^+7$ and $^+14$.

3. Find the sum of $^+(\frac{1}{2})$ and $^+(\frac{3}{4})$. Of $^+(\frac{5}{6})$ and $^+(\frac{11}{12})$. Of $^-(\frac{3}{8})$ and $^-(\frac{5}{12})$. Of $^-(\frac{1}{4})$ and $^-(\frac{15}{20})$.

Find the value of $^+a + {}^+b$,

4. When $a = 43$, $b = 63$.

5. When $a = 23$, $b = 72$.

Find the value of $^-a + {}^-b$,

6. When $a = 15$, $b = 12\frac{1}{2}$.

7. When $a = \frac{6}{7}$, $b = \frac{1}{3}$.

What is the value of $m+n$,

8. When $m = {}^+24$, $n = {}^+32$?

9. When $m = {}^-36$, $n = {}^-22$?

10. When $m = {}^+(\frac{3}{4})$, $n = {}^+(\frac{5}{8})$?

11. When $m = {}^-(\frac{2}{3})$, $n = {}^-(\frac{5}{6})$?

28. *The sum of two opposite numbers which are equal arithmetically is zero.* Or stated in symbols,

$$^+a + {}^-a \equiv 0. \tag{1}$$

Proof. Since ^-a and ^+a are *opposite* numbers *equal* in arithmetic value, they annul each other when added (§ 22).

E.g., $^-2 + {}^+2 = 0$, $^+5 + {}^-5 = 0$, $^-7 + {}^+7 = 0$, $^+8 + {}^-8 = 0$.

Ex. 1. Add $^-5$ to $^+8$.

When $^-5$ is added to $^+8$, the 5 negative units in $^-5$ annul 5 of the 8 positive units in $^+8$. There remain $8 - 5$ positive units; that is,

$$^+8 + {}^-5 = {}^+(8-5) = {}^+3.$$

Ex. 2. Add $^-9$ to $^+4$.

When $^-9$ is added to $^+4$, 4 of the 9 negative units in $^-9$ annul the 4 positive units in $^+4$. There remain $9 - 4$ negative units; that is,

$$^+4 + {}^-9 = {}^-(9-4) = {}^-5.$$

These examples illustrate the following principle:

29. To add one number to another of an opposite quality, *find the difference of their arithmetic values and prefix to it the quality-sign of the number which is arithmetically the greater.* Or, stated in symbols,

$$^+a + {}^-b \equiv {}^+(a-b), \text{ when } a > b. \tag{1}$$

$$^+a + {}^-b \equiv {}^-(b-a), \text{ when } a < b. \tag{2}$$

Proof. When $a > b$ and ^-b is added to ^+a, the b negative units in ^-b annul b of the a positive units in ^+a.

There remain $a - b$ positive units; hence, $^+(a - b)$ is the sum.

When $a < b$, a of the b negative units in ^-b annul the a positive units in ^+a. There remain $b - a$ negative units; hence, $^-(b - a)$ is the sum.

Exercise 9.

1. To make the sum zero, what number must be added to $^+3$? To $^-7$? To $^-31$? To $^+14$? To ^+a? To ^-b?

2. Find the sum of $^+8$ and $^-6$. Of $^+5$ and $^-7$. Of $^-8$ and $^+4$. Of $^+11$ and $^-15$. Of $^-5$ and $^+17$.

3. Find the sum of $^-(\frac{1}{2})$ and $^+(\frac{3}{4})$. Of $^-(\frac{5}{6})$ and $^+(\frac{11}{12})$. Of $^+(\frac{3}{8})$ and $^-(\frac{5}{12})$. Of $^+(\frac{1}{4})$ and $^-(\frac{15}{20})$.

What is the value of $^+a + {^-b}$,

4. When $a = 43$, $b = 23$?

5. When $a = 63$, $b = 43$?

6. When $a = 23$, $b = 43$?

7. When $a = 43$, $b = 63$?

8. Write six different sums each of which denotes zero.

What is the value of $x + y$,

9. When $x = {^-7}$, $y = {^+9}$?

10. When $x = {^+14}$, $y = {^-19}$?

11. When $x = {^-(\frac{7}{12})}$, $y = {^+(\frac{2}{3})}$?

12. When $x = {^+(\frac{11}{15})}$, $y = {^-(\frac{4}{5})}$?

30. The sign of continuation is $\cdots$ or ---, either of which is read, 'and so on,' or 'and so on to.'

Thus, 1, 2, 3, 4, $\cdots$, is read, '1, 2, 3, 4, and so on' indefinitely; 2, 4, 6, 8, $\cdots$ 32, is read, '2, 4, 6, 8, and so on to 32.'

The sign $\therefore$ stands for *hence* or *therefore*.

The sign $\because$ stands for *since* or *because*.

31. The integers of arithmetic number make up the series (1).

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \cdots \qquad (1)$$

Writing the positive and the negative integers in opposite directions from zero, we obtain series (2).

$$\cdots \; {}^{-}4 \quad {}^{-}3 \quad {}^{-}2 \quad {}^{-}1 \quad 0 \quad {}^{+}1 \quad {}^{+}2 \quad {}^{+}3 \quad {}^{+}4 \; \cdots \qquad (2)$$

If the divisions of the lines in (1) and (2) be taken as units of length, then each number in (1) expresses simply its *distance* from the zero point; while each number in (2) expresses not only its *distance*, but also its *direction*, from the zero point, distances to the right being regarded as positive.

Note. Arithmetic numbers are not positive numbers. An arithmetic number *has no quality.*

If to any number in series (2) we add ${}^{+}1$, we obtain the next right-hand number.

E.g., $\quad {}^{-}4 + {}^{+}1 = {}^{-}3, \quad {}^{-}2 + {}^{+}1 = {}^{-}1,$

and so on for the entire series.

Hence, if we say that a number is *increased by adding to it* ${}^{+}1$, the numbers in series (2) increase from left to right; that is,

$$\cdots, \; {}^{-}3 < {}^{-}2, \; {}^{-}2 < {}^{-}1, \; {}^{-}1 < 0, \; 0 < {}^{+}1, \; {}^{+}1 < {}^{+}2 \cdots.$$

We have, therefore, the following properties of positive and negative numbers:

(i) *Any positive number is greater than zero; while any negative number is less than zero.*

(ii) *Of two positive numbers the greater has the greater arithmetic value; while of two negative numbers the greater has the less arithmetic value.*

E.g., $\quad {}^{+}4 > 0$ by ${}^{+}4$, ${}^{-}4 < 0$ by ${}^{+}4$, ${}^{-}7 < 0$ by ${}^{+}7$,

$\quad {}^{+}4 > {}^{+}2$ by ${}^{+}2$, ${}^{-}4 < {}^{-}2$ by ${}^{+}2$, ${}^{-}7 < {}^{-}3$ by ${}^{+}4$.

NOTE. If we agreed to say that a number was increased by *adding* to it $^-1$, then the numbers in series (2) would increase from *right* to *left;* positive numbers would be less than zero, and negative numbers greater than zero. By common consent, however, it is agreed to say as above that a number is increased by adding to it $^+1$, the primary unit.

Exercise 10.

Which is the greater, and how much the greater,

1. $^+3$ or $^+7$?	4. 0 or $^+1$?	7. $^-7$ or $^+3$?
2. $^+2$ or $^-8$?	5. 0 or $^-1$?	8. $^+2$ or $^-3$?
3. $^-11$ or $^+2$?	6. $^-5$ or $^-9$?	9. $^-5$ or $^-11$?

10. When is the product of two arithmetic fractional numbers greater than each number? Less than each number? Greater than one and less than the other? Can the product of two arithmetic integral numbers ever be less than either number?

11. When is the sum of two algebraic numbers greater than each number? Less than each number? Greater than one and less than the other? Is the sum of two arithmetic numbers *always* greater than each number?

12. Multiplying by an arithmetic fractional number involves what two operations with arithmetic whole numbers? Addition of algebraic whole numbers involves the one or the other of what two operations with arithmetic numbers?

32. In proving and using identities, the following **principles concerning identical expressions** will be useful.

These principles clearly follow from the definition of identical expressions in § 15 and the axioms in § 6.

(i) *Any expression is identical with itself.*

(ii) *If each of two expressions is identical with a third, they are identical with each other.*

(iii) *If two identical expressions are added to or subtracted from two other identical expressions, the resulting expressions are identical.*

(iv) *If two identical expressions are multiplied by two other identical expressions, the products are identical.*

(v) *If two identical expressions are divided by two other identical expressions, not denoting zero, the quotients are identical.*

(vi) *If, for any expression in an identity, an identical expression is substituted, the resulting equality is an identity.*

33. The **converse** of an identity is obtained by interchanging its members; that is, the converse of $A \equiv B$ is $B \equiv A$.

If $A \equiv B$, then, from definition, $B \equiv A$.

Hence, *the proof of an identity proves its converse.*

E.g., in proving $\quad {}^+a + {}^+b \equiv {}^+(a + b)$,

we prove $\quad {}^+(a + b) \equiv {}^+a + {}^+b$.

CHAPTER III

ADDITION, SUBTRACTION, AND MULTIPLICATION OF REAL NUMBERS

34. The positive and negative numbers defined in Chapter II are together often called **real** numbers.

In performing any operation with *real* numbers, we must keep in mind that any such number is simply an *arithmetic multiple* of the *quality unit* $^+1$ or $^-1$, and that arithmetic numbers are added, subtracted, multiplied, or divided in Algebra just the same as in Arithmetic.

35. Addition. Observe that, by §§ 27 and 29, the *addition* of one real number to another is reduced to the *addition* of one *arithmetic* number to another, or to the *subtraction* of one *arithmetic* number from another.

To find the sum of three or more numbers we add the second to the first, to this sum we add the third, and so on.

$$\text{Ex. 1.}\quad {}^+8 + {}^-5 + {}^+6 + {}^-7 = {}^+3 + {}^+6 + {}^-7$$
$$= {}^+9 + {}^-7 = {}^+2.$$

$$\text{Ex. 2.}\quad {}^-7 + {}^+5 + {}^-3 + {}^+9 = {}^-2 + {}^-3 + {}^+9$$
$$= {}^-5 + {}^+9 = {}^+4.$$

36. The two following **laws of addition** are constantly used in Arithmetic and Algebra:

The commutative law. *The sum of two or more numbers is the same in whatever order they are added.*

That is, $\quad \boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} \equiv \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{a} \equiv \boldsymbol{c} + \boldsymbol{b} + \boldsymbol{a} \equiv \cdots. \qquad (A)$

Thus, we can *commute* summands (*change their order*) to suit our convenience or purpose.

E.g., in Arithmetic we write

$$\tfrac{1}{2} + 3 + \tfrac{1}{3} + 2 + \tfrac{1}{6} = \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{6} + 3 + 2. \qquad (3)$$

Here by a change of order we can add the fractions first.

Prove each of the two following particular cases of (A):

Ex. 1. $^{+}4 + ^{-}5 + ^{+}6 = ^{+}4 + ^{+}6 + ^{-}5.$ § 14

Ex. 2. $^{+}2 + ^{-}3 + ^{+}4 = ^{+}2 + ^{+}4 + ^{-}3 = ^{+}4 + ^{-}3 + ^{+}2.$

Proof of law (A). This law holds true for arithmetic numbers, as is learned in Arithmetic; hence the total number of positive units in a, b, c, etc., will be the same in whatever order these summands are written. For the same reason the total number of negative units in a, b, c, etc., will be the same in whatever order their summands are written.

Hence the sum will be the same, however we change the order of the summands; for equal numbers of opposite units always annul each other.

The associative law. *The sum of three or more numbers is the same in whatever way the successive numbers are grouped.*

That is, $$\boldsymbol{a + b + c \equiv a + (b + c)}. \qquad (B)$$

Thus we can *associate* successive summands (*group them*) to suit our convenience or purpose.

Prove each of the two following particular cases of (B):

Ex. 1. $^{+}4 + ^{-}5 + ^{+}6 = ^{+}4 + (^{-}5 + ^{+}6).$

Ex. 2. $^{-}5 + ^{+}4 + ^{-}7 = ^{-}5 + (^{+}4 + ^{-}7).$

Proof of law (B).

$$\begin{aligned} a + b + c &\equiv b + c + a && \text{by } (A) \\ &\equiv (b + c) + a && \text{by notation} \\ &\equiv a + (b + c) && \text{by } (A) \end{aligned}$$

A similar proof would apply to any other case.

The rules for addition in Arithmetic are based on the commutative and associative laws just given.

E.g., to add 45 and 23, we have

$$\begin{aligned} 45 + 23 &= 40 + 5 + 20 + 3 && \text{by notation} \\ &= 40 + 20 + 5 + 3 && \text{by } (A) \\ &= (40 + 20) + (5 + 3) && \text{by } (B) \\ &= 60 + 8 = 68. \end{aligned}$$

Writing one number under the other and then grouping the vertical columns, as we do in Arithmetic, is but a convenient way of applying laws (A) and (B).

37. Since, by the *laws* of *addition* in § 36, we can change the order of summands and group them to suit our purpose, we have the following rule for adding three or more numbers, some of which are positive and some negative:

Add all the numbers of one quality, then add all the numbers of the opposite quality, then add the two resulting sums.

Ex. $$\begin{aligned} {}^-5 + {}^+9 + {}^-11 + {}^+6 &= {}^-5 + {}^-11 + {}^+9 + {}^+6 && \text{by } (A) \\ &= {}^-16 + {}^+15 = {}^-1 && \text{by } (B) \end{aligned}$$

In practice, the rearrangement and regrouping of the summands should be done mentally and simultaneously.

Exercise 11.

By § 37 find the value of each of the following sums:

1. ${}^+19 + {}^-7 + {}^+5$.
2. ${}^-12 + {}^+9 + {}^-4$.
3. ${}^-22 + {}^+5 + {}^+7$.
4. ${}^+42 + {}^-9 + {}^-3$.
5. ${}^+4 + {}^-5 + {}^+6 + {}^-8 + {}^+7$.
6. ${}^-9 + {}^+6 + {}^-11 + {}^+12 + {}^-4$.
7. ${}^+15 + {}^-9 + {}^+7 + {}^-8 + {}^+11$.
8. ${}^-(\frac{2}{5}) + {}^+(\frac{2}{3}) + {}^-(\frac{5}{6}) + {}^+(\frac{1}{2})$.

Find the value of $x + y + z + v$.

9. When $x = {}^-25$, $y = {}^+32$, $z = {}^-45$, $v = {}^+28$.
10. When $x = {}^+94$, $y = {}^-75$, $z = {}^+82$, $v = {}^-65$.

38. From the definition of zero it follows that

$$a + 0 \equiv a.$$

That is, *any number plus zero equals the number itself.*

E.g., $7 + 0 = 7,\ 8 + 0 = 8.$

Also, $9 + (2 - 2) = 9,\ 6 + (5 - 5) = 6.$

39. Subtraction is the *inverse of addition.* Given a sum and one of its two parts, subtraction is the operation of finding the other part.

As in Arithmetic, the given sum is called the **minuend**, the given part the **subtrahend**, and the required part the **remainder.**

Hence, to *subtract* any subtrahend from any minuend is to find a third number, the remainder, which added to the subtrahend gives the minuend.

Ex. $^+9 = {}^+9 + (^+5 + {}^-5)$ §§ 28, 38

$= (^+9 + {}^+5) + {}^-5.$ § 36

Hence,

$^+9 + {}^+5$ is the number which must be added to $^-5$ to obtain $^+9$;

that is, $^+9 - {}^-5 = {}^+9 + {}^+5.$

Here the remainder $^+9 + {}^+5$ is obtained by adding to the minuend $^+9$, the subtrahend $^-5$ with its quality changed.

This example illustrates the following rule:

40. To subtract one real number from another, *add to the minuend the subtrahend with its quality changed from* $^+$ *to* $^-$, *or from* $^-$ *to* $^+$.

That is, $$M - {}^+a \equiv M + {}^-a, \quad (1)$$

$$M - {}^-a \equiv M + {}^+a, \quad (2)$$

when M is any real number.

Proof. If to the second member of (1) we add the subtrahend, ^+a, we obtain the minuend M; that is,

$$(M + {}^-a) + {}^+a \equiv M + ({}^-a + {}^+a) \qquad \text{by } (B)$$

$$\equiv M \qquad §§\ 28,\ 38$$

also, $$(M + {}^+a) + {}^-a \equiv M + ({}^+a + {}^-a) \equiv M.$$

Hence, by § 39, the second member of (1) or (2) is a remainder.

Ex. 1. $^-4 - {}^+7 = {}^-4 + {}^-7 = {}^-11.$

Ex. 2. $^-5 - {}^-8 = {}^-5 + {}^+8 = {}^+3.$

Thus, subtracting any real number gives the same result as adding its *arithmetically equal opposite number.*

E.g., subtracting \$200 credit from an estate is equivalent to adding \$200 debt; and subtracting \$300 debt is equivalent to adding \$300 credit.

Subtracting \$100 income is equivalent to adding \$100 expenditure.

Exercise 12.

Perform each of the following indicated subtractions:

1. $^+19 - {}^+7.$
2. $^-23 - {}^+12.$
3. $^-16 - {}^-30.$
4. $^+6 - {}^+7.$
5. $^+12 - {}^+20.$
6. $^-214 - {}^+25.$
7. $^-20 - {}^-25.$
8. $^-68 - {}^-98.$
9. $^-118 - {}^-120.$

What is the value of $a - b$,

10. When $a = {}^+5$, $b = {}^+4$?
11. When $a = {}^+7$, $b = {}^+9$?
12. When $a = {}^-4$, $b = {}^-7$?
13. When $a = {}^-14$, $b = {}^-11$?
14. From $^+4 + {}^-8 + {}^+9 + {}^-3$ subtract $^+7 + {}^-2 + {}^+9 + {}^-8$.
15. From $^-10 + {}^-7 + {}^+15 + {}^-3$ subtract $^+7 + {}^-11 + {}^-17$.

41. When a monomial or the first term of a polynomial is preceded by the *sign of operation* $+$ or $-$, zero is to be understood before this sign of operation.

Thus, $\quad -{}^{+}a \equiv 0 - {}^{+}a \equiv {}^{-}a, \;\; -{}^{-}a \equiv 0 - {}^{-}a \equiv {}^{+}a.$

Again, $\quad -{}^{+}5 + {}^{+}7 = 0 - {}^{+}5 + {}^{+}7 = {}^{-}5 + {}^{+}7.$

42. *Successive subtractions or successive additions and subtractions* can be performed from left to right, one at a time in succession.

Ex. 1. $\quad {}^{+}8 - {}^{-}3 - {}^{+}2 - {}^{-}6 = {}^{+}11 - {}^{+}2 - {}^{-}6$
$$= {}^{+}9 - {}^{-}6 = {}^{+}15.$$

We can, however, express each term to be subtracted, as a term to be added, and then apply the principle in § 37, for finding the sum of three or more numbers.

Ex. 2. $\quad {}^{+}8 - {}^{-}3 - {}^{+}2 - {}^{-}6 = {}^{+}8 + {}^{+}3 + {}^{-}2 + {}^{+}6 \qquad (1)$
$$= {}^{+}17 + {}^{-}2 = {}^{+}15.$$

43. Commutative law of subtraction. Since each term to be subtracted can be expressed as a term to be added, the *commutative law* holds for subtraction as well as for addition, provided *the sign of operation* + *or* − *before each term is transferred with the term itself.*

E.g., $\quad {}^{+}7 - {}^{-}8 + {}^{-}9 - {}^{+}4 = -{}^{-}8 + {}^{+}7 - {}^{+}4 + {}^{-}9$
$$= -{}^{+}4 - {}^{-}8 + {}^{-}9 + {}^{+}7.$$

Exercise 13.

Find the value of each of the following expressions:

1. ${}^{+}6 + {}^{-}2 - {}^{+}3.$ 2. ${}^{-}14 - {}^{+}9 + {}^{-}4.$ 3. ${}^{+}32 + {}^{-}5 - {}^{-}16.$

4. ${}^{+}6 - {}^{-}2 + {}^{+}3.$ 5. ${}^{+}4 - {}^{-}2 - {}^{+}3 + {}^{-}2 - {}^{+}5 + {}^{+}3 - {}^{-}6.$

6. ${}^{+}25 - {}^{+}14 + {}^{-}10 + {}^{+}14 - {}^{-}5 - {}^{+}18 + {}^{+}16 + {}^{-}18.$

7. ${}^{-}35 + {}^{-}5 - {}^{-}32 + {}^{+}24 - {}^{-}14 + {}^{-}28 - {}^{-}8.$

44. From the definition of zero it follows that

$$a - 0 \equiv a.$$

That is, *any number minus zero equals the number itself.*

45. Multiplication. As in Arithmetic, the number multiplied is called the **multiplicand**, the number which multiplies is called the **multiplier**, and the result the **product**.

In Arithmetic the *product* 9×3 is obtained by *taking the multiplicand* 9 *three times*, and the *multiplier* 3 is obtained by *taking the primary unit* 1 *three times.*

The *product* $9 \times \frac{2}{3}$ is obtained by *dividing the multiplicand* 9 *by* 3, *and multiplying the result by* 2, and the *multiplier* $\frac{2}{3}$ is obtained *by dividing the primary unit* 1 *by* 3, *and multiplying the result by* 2.

In each case we obtain the product by doing to the multiplicand just what is done to the primary unit to obtain the multiplier.

Hence, we define multiplication as follows:

To **multiply** one number by another is to do to the multiplicand just what is done to the *primary unit* to obtain the multiplier.

The multiplicand and the multiplier together are called the **factors** of the product.

46. Multiplier any arithmetic number. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be any two *arithmetic* numbers; then $\boldsymbol{b}$ *times* $\boldsymbol{a}$ *units of any kind is equal to* $\boldsymbol{ab}$ *units of that kind;* that is,

$$^{+}\boldsymbol{a} \times \boldsymbol{b} \equiv {}^{+}(\boldsymbol{ab}), \tag{1}$$

and

$$^{-}\boldsymbol{a} \times \boldsymbol{b} \equiv {}^{-}(\boldsymbol{ab}). \tag{2}$$

E.g., $\quad {}^{+}4 \times 5 = {}^{+}20,\ {}^{-}7 \times 4 = {}^{-}28,\ {}^{-}(\frac{3}{2}) \times 8 = {}^{-}12.$

47. Multiplier any positive or any negative number.

To obtain $^{+}\boldsymbol{b}$ from the *primary unit* $^{+}1$ we take that unit $\boldsymbol{b}$ times; hence, by definition, *to multiply any number by* $^{+}\boldsymbol{b}$ *we take that number* $\boldsymbol{b}$ *times;*

that is,

$$^{+}\boldsymbol{a} \times {}^{+}\boldsymbol{b} \equiv {}^{+}a \times b \equiv {}^{+}(\boldsymbol{ab}), \tag{1}$$

and

$$^{-}\boldsymbol{a} \times {}^{+}\boldsymbol{b} \equiv {}^{-}a \times b \equiv {}^{-}(\boldsymbol{ab}). \tag{2}$$

Hence, to multiply any number by $^{+}1$ is to take that number once; that is, $^{+}a \times {}^{+}\mathbf{1} \equiv {}^{+}a$; $^{-}a \times {}^{+}\mathbf{1} \equiv {}^{-}a$.

To obtain $^{-}\boldsymbol{b}$ from the *primary unit* $^{+}1$, we change the quality of that unit and multiply the result by $\boldsymbol{b}$; hence, *to multiply any number by* $^{-}\boldsymbol{b}$, *we change the quality of that number and multiply the result by* $\boldsymbol{b}$;

that is, $$^{+}\boldsymbol{a} \times {}^{-}\boldsymbol{b} \equiv {}^{-}a \times b \equiv {}^{-}(\boldsymbol{ab}), \qquad (3)$$

and $$^{-}\boldsymbol{a} \times {}^{-}\boldsymbol{b} \equiv {}^{+}a \times b \equiv {}^{+}(\boldsymbol{ab}). \qquad (4)$$

Hence, to multiply any number by $^{-}1$ is to change the quality of that number; that is, $^{+}a \times {}^{-}1 \equiv {}^{-}a$; $^{-}a \times {}^{-}1 \equiv {}^{+}a$.

From identities (1) and (4) we have the law,

The product of two real numbers like in quality is positive.

From identities (2) and (3) we have the law,

The product of two numbers opposite in quality is negative.

These two laws together are called the **law of quality** of products.

From identities (1), (2), (3), (4), it follows that

The arithmetic value of the product of two real numbers is the product of their arithmetic values.

E.g., $^{+}5 \times {}^{+}7 = {}^{+}35$, $\quad$ $^{-}6 \times {}^{-}8 = {}^{+}48$.

$^{+}4 \times {}^{-}9 = {}^{-}36$, $\quad$ $^{-}7 \times {}^{+}8 = {}^{-}56$.

Exercise 14.

Find the value of each of the following numeral expressions:

1. $^{+}2 \times {}^{+}4$. 3. $^{+}9 \times {}^{-}8$. 5. $^{-}21 \times {}^{+}3$. 7. $^{+}22 \times {}^{-}6$.

2. $^{-}2 \times {}^{-}7$. 4. $^{-}11 \times {}^{-}1$. 6. $^{+}31 \times {}^{-}1$. 8. $^{-}32 \times {}^{-}4$.

9. $^{+}10 \times {}^{-}3 + {}^{-}8 \times {}^{+}2$. 11. $^{-}6 \times {}^{-}5 + {}^{+}8 \times {}^{-}4 - {}^{+}12 \times {}^{-}5$.

10. $^{+}14 \times {}^{-}2 + {}^{-}6 \times {}^{-}5$. 12. $^{-}9 \times {}^{+}2 + {}^{+}16 \times {}^{-}4 - {}^{-}14 \times {}^{+}3$.

When $a = {}^{+}2$, $b = {}^{-}4$, $m = {}^{-}3$, $n = {}^{+}9$, $x = {}^{+}6$, find the value of each of the following literal expressions:

13. $ab + mx$.	**16.** $ax - nb$.	**19.** $(a+b)(n+m)$.
14. $ax + bm$.	**17.** $(a-b)x$.	**20.** $(a-b)(n-m)$.
15. $am - bx$.	**18.** $(m-n)b$.	**21.** $(b-x)(m-n)$.

48. Continued products. By § 47, we obtain

$$^{+}a \times {}^{+}b \times {}^{+}c \equiv {}^{+}(ab) \times {}^{+}c \equiv {}^{+}(abc).$$

$$^{+}a \times {}^{+}b \times {}^{-}c \equiv {}^{+}(ab) \times {}^{-}c \equiv {}^{-}(abc).$$

$$^{+}a \times {}^{-}b \times {}^{-}c \equiv {}^{-}(ab) \times {}^{-}c \equiv {}^{+}(abc).$$

$$^{-}a \times {}^{-}b \times {}^{-}c \equiv {}^{+}(ab) \times {}^{-}c \equiv {}^{-}(abc).$$

From these and similar identities we have the following laws which are more general than those in § 47:

A product which contains an odd number of negative factors is negative; any other product is positive.

The arithmetic value of a product is the product of the arithmetic values of its factors.

Ex. Find the value of ${}^{+}3 \times {}^{-}2 \times {}^{+}4 \times {}^{-}6 \times {}^{-}5$.

The product is *negative*, since there is an *odd* number, 3, of negative factors; its arithmetic value is $3 \times 2 \times 4 \times 6 \times 5$, or 720.

Hence, $${}^{+}3 \times {}^{-}2 \times {}^{+}4 \times {}^{-}6 \times {}^{-}5 = {}^{-}720.$$

Exercise 15.

When $a = {}^{-}2$, $b = {}^{+}4$, $c = {}^{-}6$, $x = {}^{-}3$, $y = {}^{-}5$, find the value of each of the following literal expressions:

1. abc.	**4.** $(b+a)cx$.	**7.** $axy - bc$.
2. $abxy$.	**5.** $(x-y)abc$.	**8.** $xy - abc$.
3. $abcxy$.	**6.** $(b+c)axy$.	**9.** $x + abcy$.

10. Prove $^{+}1 \times {}^{-}1 \times {}^{-}1 \times {}^{-}1 = {}^{-}1$; $^{-}1 \times {}^{-}1 \times {}^{-}1 \times {}^{-}1 = {}^{+}1$.

11. Prove $^{+}a \times {}^{-}b \times {}^{-}c \equiv ({}^{+}1 \times {}^{-}1 \times {}^{-}1)(abc)$.

12. Prove $^{-}a \times {}^{+}b \times {}^{-}c \times {}^{-}x \equiv ({}^{-}1 \times {}^{+}1 \times {}^{-}1 \times {}^{-}1)(abcx)$.

Examples 11 and 12 illustrate that *the product of two or more numbers is equal to the product of their quality-units multiplied by the product of their arithmetic values.*

49. The two following laws of multiplication are constantly used in Arithmetic and Algebra:

The commutative law. *The product of two or more numbers is the same in whatever order the factors are multiplied.*

That is, $$\boldsymbol{abc \equiv acb \equiv cba} \equiv \cdots. \qquad (A')$$

Prove each of the two following particular cases of (A'):

Ex. 1. $^{+}2 \times {}^{-}3 \times {}^{+}4 \times {}^{-}5 = {}^{-}5 \times {}^{-}3 \times {}^{+}2 \times {}^{+}4$.

Ex. 2. $^{-}3 \times {}^{+}7 \times {}^{-}2 \times {}^{-}1 = {}^{-}2 \times {}^{+}7 \times {}^{-}1 \times {}^{-}3$.

Proof. In Arithmetic we have learned that this law holds true for arithmetic numbers. Hence, by § 48, the arithmetic value of a product of real numbers is the same in whatever order the factors are multiplied.

From the *law of quality*, in § 48, it follows that the *quality* of a product of real numbers will be the same in whatever order the factors are multiplied.

Hence, a change of order of factors affects neither the arithmetic value nor the quality of their product.

The associative law. *The product of three or more numbers is the same in whatever way the successive factors are grouped.*

That is, $$\boldsymbol{abc \equiv a(bc)}. \qquad (B')$$

Prove each of the following particular cases of (B'):

Ex. 1. $^{-}3 \times {}^{+}4 \times {}^{-}2 = {}^{-}3 \times ({}^{+}4 \times {}^{-}2)$.

Ex. 2. $^{+}5 \times {}^{-}6 \times {}^{-}1 \times {}^{+}2 = {}^{+}5 \times ({}^{-}6 \times {}^{-}1 \times {}^{+}2)$.

Proof. $abc \equiv bca$ by (A')

$\equiv (bc)a$ by notation

$\equiv a(bc)$ by (A')

Exercise 16.

By using the commutative and associative laws, find in the *simplest* way the value of each of the following expressions:

1. ${}^{+}33 \times {}^{-}2\frac{1}{2} \times {}^{-}4$.
2. ${}^{-}123 \times {}^{-}33\frac{1}{3} \times {}^{+}3$.
3. ${}^{+}142 \times {}^{-}12\frac{1}{2} \times {}^{-}8$.
4. ${}^{+}144 \times {}^{-}3 \times {}^{-}16\frac{2}{3}$.
5. ${}^{-}37\frac{1}{2} \times {}^{-}7 \times {}^{-}4$.
6. ${}^{-}333\frac{1}{3} \times {}^{-}5 \times {}^{-}7 \times {}^{+}3$.

50. *A product of two or more factors is multiplied by a number if any one of the factors is multiplied by that number.*

Proof. $(ab) \times c \equiv (ac)b \equiv a(bc)$. § 49

51. Powers. A product of two or more equal factors is called a **power.** Any number also is often called the *first power* of itself.

E.g., the product aa is called the *second power* of a.

The product bbb is called the *third power* of b.

aa is written a^2; aaa is written a^3;

$aaa \cdots$ to n factors, written a^n, is read '*the* nth *power of* a.'

a^2 is often read '*the square of* a,' and a^3 '*the cube of* a.'

In a^n, a is called a **base.** Thus in 3^4, 3 is a base; in x^3, x is a base; in y^m, y is a base.

52. A **positive integral exponent** is a whole number which (written to the right and a little above a base) indicates how many times the base is used as a factor, as 3 in a^3, or n in a^n.

To avoid ambiguity, a base which is not denoted by a single symbol must be enclosed within parentheses:

E.g., $(^-3)^2 = {}^-3 \times {}^-3 = {}^+9$, while $^-3^2 = {}^-1\,(3 \times 3) = {}^-9$.

Again, $(4 \times 5)^2 = (4 \times 5)\,(4 \times 5) = 20 \times 20 = 400,$

while $4 \times 5^2 = 4 \times (5 \times 5) = 4 \times 25 = 100.$

The meaning of fractional and negative exponents will be determined in a later chapter.

A power is said to be **odd** or **even** according as its exponent is *odd* or *even*.

53. Quality of a power. An *odd* power of a *negative* base is the only power which involves an *odd* number of *negative* factors; hence, by the law of quality in § 48 it follows that

An odd power of a negative base is negative, and an even power positive; any power of a positive base is positive.

E.g., any power of $^+1$ is $^+1$; any even power of $^-1$ is $^+1$; any odd power of $^-1$ is $^-1$.

Exercise 17.

1. What number is the *base* and what the exponent in $^-3^4$? In $(^-8)^3$? In $3\,xy^n$? In $(3\,xy)^n$? In $(a+b)^n$? In $a+b^n$? In $(x^2)^4$?

Find the value of each of the following expressions:

2. $^-3^4$. 3. $(^-3)^4$. 4. $17-3^2$. 5. $(17-3)^2$.

Express each of the following products by a base and exponent:

6. $(xy)\,(xy)\,(xy) \cdots$ to 8 factors.

7. $(a+b)\,(a+b)\,(a+b) \cdots$ to 12 factors.

Express in symbols:

8. The sum of the cubes of x and y.

9. The cube of the sum of x and y.

10. The sum of the squares of a, b, and c.

11. The square of the sum of a, b, and c.

54. If, in any one of the identities in § 48, the quality of one factor is changed, the quality of the product is changed, but its arithmetic value remains the same.

This illustrates the following principle:

The quality of any product is changed by changing the quality of one, or of any odd number, of its factors.

Proof. By changing the quality of an odd number of factors, the number of negative factors in the product is changed from odd to even, or from even to odd; hence, by § 48, the quality of the product is changed.

NOTE. When for brevity we speak of the quality of an expression, we mean, of course, the quality of the number which the expression denotes.

55. *The quality of an expression is changed by changing the quality of each of its terms.*

Proof. Changing the quality of a term does not affect its arithmetic value. Hence, changing the quality of each term of an expression will simply change a positive sum into an arithmetically equal negative sum, or *vice versa*.

This principle is illustrated by the fact that if in a business account we change debts into credits, and credits into debts, the balance will not be changed in amount, but it will be changed from credits to debts, or from debts to credits.

Ex. 1. Change in four ways the quality of $^-4 \times {}^+3 \times {}^-2$. Of $^-3 \times {}^-5 \times {}^-7$. Of $^+a \times {}^+b \times {}^-c$. Of $^+a \times {}^-b \times {}^-c$. Of $^+x \times {}^+y \times {}^-2$.

Ex. 2. Change in two ways the quailty of $^-4 \times {}^+3 - {}^-2 \times {}^+7$. Of $^+a \times {}^-b + {}^-c \times {}^+x$. Of $^-a \times {}^-x - {}^+b \times {}^+c$.

56. Two uses of the signs + and −. Hereafter the larger signs + and − will be used, not only as *signs of operation*, but also with *numerals* as *signs of quality*.

To avoid ambiguity, parentheses will be used when needed.

Thus, in the expression,

$$(+4)-(+7)+(-3)-(-4),$$

each sign within parentheses denotes quality, and each without denotes an operation.

Again, $(-3)ax-(+4)by+(-5)cz \equiv {}^{-}3\,ax - {}^{+}4\,by + {}^{-}5\,cz.$

A letter with the small sign $^{+}$ or $^{-}$ will continue to be used to denote a general positive or a general negative number.

57. Abbreviated notation. The sign − is never omitted. But, for the sake of brevity, the sign + has been omitted, and is to be understood in the two following cases:

(i) When *no* sign is written before a monomial or before the first term of a polynomial, the sign + is to be understood.

(ii) When *only one* sign is written between two successive terms of a polynomial, the sign + is to be understood either as a sign of operation or as a sign of quality.

E.g., 2 denotes $+2$, $3\,a$ denotes $+3\,a$, and a denotes $+1\,a$.

Again, $6-5$ denotes the difference $(+6)-(+5)$ or the sum $(+6)+(-5)$; in each case the sign + is understood between 6 and 5; in the *first* case as a *sign of quality*, and in the *second* case as a *sign of operation.*

Since $\qquad (+6)-(+5)=(+6)+(-5),$

$6-5$ denotes the same number whether it is regarded as expressing the difference $(+6)-(+5)$ or the sum $(+6)+(-5)$.

Again,

$$7-5+8=(+7)-(+5)+(+8), \text{ or } (+7)+(-5)+(+8),$$

according as we regard the *written* signs in the first expression as signs of operation or as signs of quality.

Hence, in the *abridged* notation, the written signs in any polynomial can be regarded either as signs of operation or as *signs of quality.*

When all the written signs are regarded as signs of quality any polynomial becomes a *sum.*

E.g., $$-5+3-2=(-5)+(+3)+(-2),$$

or the sum of the terms -5, $+3$, and -2.

Again, $$7\,ac-4\,x+3\,y\equiv+7\,ac+(-4)x+(+3)y,$$

or the sum of the terms $+7\,ac$, $-4\,x$, and $+3\,y$.

In general formulas, such as (A), (B), etc., it is usually better to regard the written signs as signs of operation; but in most other cases it is preferable to regard the written signs as signs of quality and, therefore, to regard every *polynomial* as a *sum.*

58. **Coefficients.** If a term is resolved into two factors, either factor is called the **coefficient**, or the **co-factor**, of the other.

E.g., in $4\,abc$, $+4$ is the coefficient of abc, $+4\,a$ of bc, $+4\,ab$ of c, abc of $+4$, and ba of $+4\,c$.

A **numeral coefficient** is a coefficient expressed entirely by numerals, and a sign of quality written or understood.

A **literal coefficient** is a coefficient which involves one or more letters.

E.g., in $-4\,xy$, -4 is the *numeral* coefficient of xy; x is the *literal* coefficient of $-4\,y$, y of $-4\,x$, and $-4\,x$ of y.

When in a term no numeral factor is written, 1 is understood, *e.g.*, a denotes $+1\cdot a$ and $-a$ denotes $-1\cdot a$; abc denotes $+1\cdot abc$ and $-abc$ denotes $-1\cdot abc$.

Exercise 18.

Find the value of each of the following expressions:

1. $15 - 9$.
2. $-9 + 7$.
3. $-8 - 6$.
4. $(-3)(-4)$
5. $(-11) \times 7$.
6. $(-7) \div (-4)$.
7. $9 - 7 + 4 - 3 + 5$.
8. $18 \div (-3) \times (-4) \div 8$.
9. $35 \div (-7) \times 6 \div 15 \times (-2)$.

Find the value of $a + b - c + d$ and $a - (-b + c - d)$.

10. When $a = 2$, $b = -4$, $c = -6$, $d = -7$.
11. When $a = -7$, $b = -8$, $c = 5$, $d = -6$.

Find the value of $x(y - v + z)$.

12. When $x = 5$, $y = -7$, $v = -9$, $z = 8$.
13. When $x = -5$, $y = 10$, $v = -4$, $z = -7$.

Find the value of $x \div (y - v - z)$.

14. When $x = -10$, $y = -6$, $v = -9$, $z = 8$.
15. When $x = -16$, $y = -10$, $v = -12$, $z = 6$.
16. What is the coefficient of a in a? In $-a$? In $-7ay$?
17. In the expression $-8ab(x - y)$, what is the coefficient of $x - y$? Of $b(x - y)$? Of $8a$? Of $-8(x - y)$?
18. If the sum $(x - y) + (x - y) + (x - y) + \cdots$ to a summands is expressed as a product, what is the coefficient of $x - y$?

59. Having given a product and one factor, **division** is the operation of finding the other factor. That is, if n is one factor of m, $m \div n$ denotes the other factor; whence

$$(m \div n) \times n \equiv m. \qquad (1)$$

60. The distributive law. *The product of a polynomial by a monomial is equal to the sum of the products obtained by multiplying each term of the polynomial by the monomial; and conversely.*

That is, $$(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\cdots)\boldsymbol{x} \equiv \boldsymbol{ax}+\boldsymbol{bx}+\boldsymbol{cx}+\cdots \qquad (C)$$

The distributive law lies at the basis of multiplication in Arithmetic, *e.g.*, if we wish to multiply any number as 248 by 7, we separate 248 into the parts 200, 40, and 8, multiply each of these parts by 7 and add the results.

Thus,
$$248 \times 7 = (200+40+8) \times 7 \qquad (1)$$
$$= 200 \times 7 + 40 \times 7 + 8 \times 7 \qquad (2)$$
$$= 1400 + 280 + 56 = 1736.$$

We pass from (1) to (2) by the distributive law (C).

Prove each of the following particular cases of (C):

Ex. 1. $(4-3+5)\cdot(-2) = 4(-2)+(-3)\cdot(-2)+5(-2)$.

Ex. 2. $(-4+2-6)(-3) = (-4)\cdot(-3)+2(-3)+(-6)\cdot(-3)$.

Ex. 3. $(a+b+c)\cdot 3 \equiv 3a+3b+3c$.

Proof. Let the multiplicand be any binomial $a+b$.

The proof involves three cases: when the multiplier is (i) a positive integer, (ii) a positive fractional number, (iii) a negative number.

(i) Let m be any positive whole number; then

$$(a+b)m \equiv (a+b)+(a+b)+\cdots \text{ to } m \text{ summands} \qquad \S\ 47$$
$$\equiv (a+a+\cdots \text{ to } m \text{ summands})$$
$$+(b+b+\cdots \text{ to } m \text{ summands}) \qquad \S\ 36$$
$$\equiv am+bm. \qquad (1)$$

(ii) Let m and n be any positive whole numbers other than zero; then $\frac{m}{n}$ will denote any positive fractional number.

$$(a+b)(m \div n)n \equiv (a+b)m \qquad \S\S\ 49,\ 59$$

$$\equiv am + bm. \qquad \text{by (1)}$$

$$\equiv a(m \div n)n + b(m \div n)n \qquad \S\S\ 49,\ 59$$

$$\equiv [a(m \div n) + b(m \div n)]n. \qquad \text{by (1)}$$

Dividing the first and last expressions by n, by (v) of § 32 we obtain

$$(a+b)(m \div n) \equiv a(m \div n) + b(m \div n). \tag{2}$$

Let r be any positive number, whole or fractional; then, from (1) and (2) we have

$$(a+b)r \equiv ar + br. \tag{3}$$

(iii) If the quality of equal numbers is changed from $+$ to $-$, or from $-$ to $+$, the resulting numbers will be equal.

Hence, changing the quality of both members of (3) we have

$$(a+b)(-r) \equiv a(-r) + b(-r), \qquad \S\S\ 54,\ 55$$

where $-r$ is any negative number, whole or fractional.

A similar proof would apply to any polynomial as well as to $a+b$; hence the law as stated in (C).

Ex. 1. Multiply $3a^2 - 5a + 3b$ by $2x$.

$$(3a^2 - 5a + 3b)(2x) \equiv (3a^2)(2x) + (-5a)(2x) + (3b)(2x)$$

$$\equiv 6a^2x - 10ax + 6bx.$$

Observe that in applying (C) we regard a polynomial as a *sum*.

Ex. 2. Multiply $2x^3 - 3x^2 - 2x$ by $-3a$.

$$(2x^3 - 3x^2 - 2x)(-3a) \equiv (2x^3)(-3a) + (-3x^2)(-3a) + (-2x)(-3a)$$

$$\equiv -6ax^3 + 9ax^2 + 6ax.$$

Exercise 19.

Multiply:

1. $x+2$ by 3.
2. $6a-7b$ by -2.
3. $2x-6$ by -5.
4. $2x-b$ by $-3a$.
5. $ax-3b$ by $-2c$.
6. $2cy-4x$ by $-a$.
7. $2a-3b-c$ by $-2x$.
8. $-3x+2y-5z$ by $3a$.
9. x^2-3x+4 by $-2a$.
10. $x^3-2y-3z$ by $-5a$.
11. $-2x^2+3xy-4y^2-x+2y-7$ by $-3a$.

CHAPTER IV

ADDITION AND SUBTRACTION OF INTEGRAL LITERAL EXPRESSIONS

61. An **integral literal expression** is an expression which involves only additions, subtractions, multiplications, and positive integral powers of its *letters.*

Any expression which contains a literal divisor is called a **fractional literal expression.**

E.g., $a^2 + \frac{3}{4}$ and $4x^2 - \frac{2}{3}b^4$ are *integral* literal expressions; while $\frac{x}{y}$ and $\frac{8}{4-b}$ are *fractional* literal expressions.

A letter can, in general, denote any *integral* or *fractional* number; hence, any literal expression can have any integral or fractional value.

E.g., when $x = \frac{1}{2}$ and $y = \frac{1}{3}$, the *integral literal* expression

$$x + y = \tfrac{1}{2} + \tfrac{1}{3} = \tfrac{5}{6}, \text{ a } \textit{fractional} \text{ number.}$$

Also, when $x = 2$ and $y = 3$, the integral expression $\frac{2}{3}xy = \frac{20}{3}$.

Again, when $x = 10$ and $y = 2$, the *fractional* expression $\frac{x}{y} = 5$.

The pupil must clearly distinguish between integral and fractional *expressions* and integral and fractional *numbers.*

62. Like or **similar terms** are terms which do not differ, or which differ only in their coefficients.

E.g., $4ab$ and $4ab$ are like terms; so also are $4ab$ and $-10ab$. Again, $6axy$ and $-4bxy$ are similar terms, if we regard $6a$ and $-4b$, respectively, as the coefficients of xy in the two terms; but if 6 and -4 be taken as the coefficients, these terms are *dissimilar.*

63. Sum of similar terms. The converse of the distributive law in § 60 is

$$ax + bx + cx + \cdots \equiv (a + b + c + \cdots)x. \qquad (C)$$

That is, *the sum of two or more similar terms is equal to the sum of their coefficients into their common factor.*

1. Find the sum of $7a$, $-5a$, $4a$.

$$(+7)a + (-5)a + (+4)a \equiv (7 - 5 + 4)a \equiv 6a.$$

2. Find the sum of $3ab^2$, $-5ab^2$, $-8ab^2$.

$$(+3)ab^2 + (-5)ab^2 + (-8)ab^2 \equiv (3 - 5 - 8)ab^2 \equiv -10ab^2.$$

3. Find the sum of $7(a-b)$, $-5(a-b)$, $4(a-b)$.

$$(+7)(a-b) + (-5)(a-b) + (+4)(a-b) \equiv (7-5+4)(a-b) \equiv 6(a-b).$$

64. By § 57 the sum of two or more terms is *indicated* by writing them in succession, each term being preceded by the *sign of quality* of its *numeral* coefficient.

The sum of unlike terms can only be *indicated.*

E.g., the sum of $-5c$, $7a$, and $-9b$ is

$$-5c + 7a - 9b, \text{ or } 7a - 5c - 9b.$$

Again, the sum of $-3ax$, $-5by$, and $6cz$ is

$$-3ax - 5by + 6cz, \text{ or } 6cz - 3ax - 5by.$$

Exercise 20.

Find the sum of:

1. $2a$, $-3a$, $5a$.
2. $-4x$, $2x$, $-x$.
3. ab, $-2ab$, $3ab$.
4. $2a^5$, $-3a^5$, $7a^5$.
5. x^n, $-2x^n$, $4x^n$.
6. $4ab^2$, $-7ab^2$, $3ab^2$.
7. $-3x^n$, $5x^n$, $-4x^n$.
8. $2ac^2$, $-5ac^2$, $-8ac^2$.
9. $-5a^2c^3$, $-3a^2c^3$, $9a^2c^3$.
10. $4b^ny^m$, $-7b^ny^m$, $9b^ny^m$.

11. $7\,ax^2,\ -5\,ax^2,\ 4\,ax^2,\ -9\,ax^2,\ -14\,ax^2,\ 25\,ax^2.$

12 $9\,ax^2,\ -ax^2,\ 4\,ax^2,\ -7\,bx^2,\ -14\,cx^2.$

13. $-xyz,\ \frac{3}{2}\,xyz,\ -\frac{5}{6}\,xyz,\ 6\,xyz,\ -\frac{2}{3}\,xyz,\ \frac{11}{6}\,xyz.$

14. $(x-a)^2,\ -2(x-a)^2,\ 4(x-a)^2,\ -5(x-a)^2,\ 12(x-a)^2.$

15. $(x^2+y^2),\ -5(x^2+y^2),\ 9(x^2+y^2),\ -3(x^2+y^2),\ -7(x^2+y^2).$

16. $(x^3-y^3),\ -4(x^3-y^3),\ -3(x^3-y^3),\ -7(x^3-y^3),\ 8(x^3-y^3).$

Simplify each of the following expressions by combining like terms:

17. $x^2-7\,x^2+4\,x^2-5\,x^2.$

18. $x^n-5\,x^n+3\,x^n-7\,x^n.$

19. $x^2y^2-8\,x^2y^2+4\,x^2y^2-7\,x^2y^2.$

20. $ax^2-7\,x^2+bx^2-5\,x^2.$

21. $-9\,x^2+17\,x^2+3\,x^2-4\,x^2-x^2+y^2-5\,y^2.$

22. $3\,ab^2-7\,ab^2+8\,ab^2-4\,ab^2+7\,cx^2-11\,cx^2.$

23. $-12\,x^3+4\,x^3-9\,x^3+7\,x^3+8\,a^3-9\,a^3+7\,a^3.$

24. $7\,abcd-11\,abcd+41\,abcd+7\,xy-20\,xy.$

25. $-\frac{5}{2}\,x^2-2\,x^2+\frac{5}{3}\,x^2+8\,y^2-\frac{3}{4}\,y^2-\frac{5}{3}\,y^2.$

26. $7\,x^2+2\,a^2-5\,x^2-3\,a^2.$

$$7\,x^2+2\,a^2-5\,x^2-3\,a^2 \equiv 7\,x^2-5\,x^2+2\,a^2-3\,a^2 \quad \text{by } (A)$$
$$\equiv 2\,x^2-a^2.$$

27. $7\,ab-5\,xy+3\,ab+2\,xy-6\,ab-xy.$

28. $-9\,ax^2+5\,by^2+7\,ax^2-3\,by^2+11\,ax^2+4\,by^2.$

29. $-7\,cy^2-4\,ab+9\,xz+11\,cy^2+10\,ab-5\,xz-ab.$

30. $2(x^2-1)+3(a^2+1)-4(x^2-1)-5(a^2+1).$

31. $3(a^2+b^2)-4(x+y)-7(a^2+b^2)+5(x+y).$

32. Review this exercise, solving each example mentally.

65. Addition of polynomials.

Ex. 1. Add $-3x^2+7x$ to $5x^2-4x$.

$$(5x^2-4x)+(-3x^2+7x) \equiv 5x^2-4x-3x^2+7x \quad \text{by converse of } (B)$$
$$\equiv 2x^2+3x. \quad \text{by } (A), (B)$$

Ex. 2. Find the sum of

$$4x^2-3xy+y^2,\ -2x^2-5xy-6y^2, \text{ and } 2xy-x^2-3b^2.$$

In adding polynomials, it is convenient to write them under each other, placing like terms in the same column.

Thus, $(4x^2-3xy+y^2)+(-2x^2-5xy-6y^2)+(2xy-x^2-3b^2)$

can be written
$$\begin{array}{rrrr} 4x^2 & -3xy & +y^2 & \\ -2x^2 & -5xy & -6y^2 & \\ -x^2 & +2xy & & -3b^2 \\ \hline x^2 & -6xy & -5y^2 & -3b^2. \end{array}$$

Here the rows of terms are the groups of terms as given, while the columns of terms are the groups of similar terms obtained by rearranging and regrouping by laws (A) and (B).

Since there is no carrying as in Arithmetic, the addition can be performed from left to right, or from right to left.

66. When in a polynomial the exponents of some one letter increase or decrease, from term to term, the polynomial is said to be *arranged in ascending*, or in *descending*, powers of that letter.

This letter is called the *letter of arrangement*.

E.g., the polynomial $x^3+2x^2y+3xy^2+4y^3$ is arranged

in *descending* powers of x, x being the letter of arrangement;

or, in *ascending* powers of y, y being the letter of arrangement.

In arranging a polynomial in ascending or descending powers of any letter, we must first combine all the terms which contain the same power of that letter.

In adding polynomials, it is usually convenient to arrange them in ascending, or descending, powers of some letter, as below:

Ex. 1. Find the sum of $2x^2 - 3x^3 + x - 4$,

$7x - 4x^2 + 5x^3 + 5$, and $7x^2 - 4x^3 + 2x - 1$.

Arranging each polynomial in descending powers of x, we have

$$\begin{array}{r}
-3x^3 + 2x^2 + x - 4 \\
5x^3 - 4x^2 + 7x + 5 \\
-4x^3 + 7x^2 + 2x - 1 \\
\hline
-2x^3 + 5x^2 + 10x
\end{array}$$

Exercise 21.

Find the sum of:

1. $a + 2b - 3c,\ -3a + b + 2c,\ 2a - 3b + c.$

2. $-3x + 2y + z,\ x - 3y + 2z,\ 2x + y - 3z.$

3. $-15a - 19b - 18c,\ 14a + 15b + 8c,\ a + 5b + 9c.$

4. $5ax - 7by + cz,\ ax + 2by - cz,\ -3ax + 2by + 3cz.$

5. $20p + q - r,\ p - 20q + r,\ p + q - 20r.$

6. $-5ab + 6bc - 7ac,\ 8ab - 4bc + 3ac,\ -2ab - 2bc + 4ac.$

7. $pq + qr - pr,\ -pq + qr + pr,\ pq - qr + pr.$

8. $2ab + 3ac + 6abc,\ -5ab + 2bc - 5abc,\ 3ab - 2bc - 3ac.$

9. $x^2 + xy - y^2,\ -z^2 + yz + y^2,\ xz + z^2 - x^2.$

10. $5a^3 - 3c^3 + d^3,\ b^3 - 2a^3 + 3d^3,\ 4c^3 - 2a^3 - 3d^3.$

11. $x^2 + y^2 - 2xy,\ 2z^2 - 3y^2 - 4yz,\ 2x^2 - 2z^2 - 3xz.$

12. $x^3 + 3x^2y + 3xy^2,\ -3x^2y - 6xy^2 - x^3,\ 3x^2y + 4xy^2.$

13. $x^5 - 4x^4y - 5x^3y^3,\ 3x^4y + 2x^3y^3 - 6xy^4,\ 3x^3y^3 + 6xy^4 - y^5.$

14. $a^3 - 4a^2b + 6abc,\ a^2b - 10abc + c^3,\ b^3 + 3a^2b + abc.$

15. $3a^2 - 10b^2 + 5c^2 - 7bc,\quad -a^2 + 4b^2 - 10c^2 + 3ab,\ c^2 + 11bc + 8ac - 2ab,\ 4c^2 - 4bc + ac,\ -2a^2 + 6b^2 - 9ac - bc.$

16. $4x^5+12x^3-x-10,\ 11x^2-2x^4-x^5+9,\ 9x^2-3x^5+4x,\ 4x^3-x^4-5,\ 6x^4-x^3+2x^2-7.$

17. $\frac{1}{2}x^2-\frac{1}{3}x+\frac{1}{4},\ -\frac{3}{4}x^2+\frac{1}{4}x-\frac{3}{4},\ \frac{5}{8}x^2+\frac{2}{3}x+\frac{1}{2}.$

18. $\frac{2}{3}a^2+\frac{1}{3}ab-\frac{1}{4}b^2,\ \frac{2}{3}a^2-ab-\frac{5}{4}b^2,\ -a^2-\frac{2}{3}ab+2b^2.$

19. $-\frac{2}{3}x^2-xy+y^2,\ 3x^2-\frac{2}{5}xy-\frac{1}{2}y^2,\ -\frac{3}{2}x^2+2xy-\frac{2}{3}y^2.$

20. $-\frac{3}{2}x^3-\frac{3}{4}xy^2+2y^3,\ \frac{3}{2}x^2y+xy^2+\frac{1}{2}y^3,\ \frac{1}{2}x^3-2x^2y-\frac{3}{2}y^3.$

21. $x^3-3ax^2+5a^2x-a^3,\ 2x^3+4ax^2-6a^2x,\ 6ax^2-3a^2x+a^3,\ -2x^3+4a^2x-5a^3.$

22. $3x^2+y^2-3yz-z^2,\ 2xy-3y^2+3yz,\ -4x^2-2xy+y^2+z^2.$

23. Given $x=b+2c-3a,\ y=c+2a-3b$, and $z=a+2b-3c$; show that $x+y+z=0$.

24. Given $a=5x-3y-2z,\ b=5y-3z-2x$, and $c=5z-3x-2y$; show that $a+b+c=0$.

67. To subtract one expression from another, *change the sign before each term of the subtrahend from + to − or from − to +, and add the result to the minuend.*

Proof. Changing the sign before each term of the subtrahend *changes* the quality of the subtrahend (§ 55); and by § 40 the minuend plus the subtrahend with its quality *changed* is equal to the remainder.

Ex. 1. From $-5x^2y$ take $4x^2y$.

$$\begin{aligned}-5x^2y-(+4x^2y)&\equiv -5x^2y+(-4x^2y)\\ &\equiv -9x^2y.\end{aligned}$$

Ex. 2. From $5x^2+xy-m$ take $2x^2+8xy-7y^2$.

Changing the sign before each term of the subtrahend from + to − or from − to +, and adding the result to the minuend, we have

$$\begin{array}{llll} 5x^2 & +\ xy & -\ m & \\ -2x^2 & -\ 8xy & & +\ 7y^2 \\ \hline 3x^2 & -\ 7xy & -\ m & +\ 7y^2, \text{ Remainder.}\end{array}$$

NOTE. The signs of the subtrahend need not be *actually* changed; the operation of changing the signs ought usually to be performed *mentally*, as in the following example.

Ex. 3. From $2x^4 - 3x^2 + 7x - 8$ take $x^4 - 2x^3 - 9x + 4$.

$$\begin{array}{r}
2x^4 \qquad\quad - 3x^2 + \ 7x - \ 8 \\
x^4 - 2x^3 \qquad\qquad - \ 9x + \ 4 \\
\hline
x^4 + 2x^3 - 3x^2 + 16x - 12
\end{array}$$

Exercise 22.

1. From $4a - 3b + c$ subtract $2a - 3b - c$.
2. From $15x + 10y - 18z$ subtract $2x - 8y + z$.
3. From $-10bc + ab - 4cd$ take $-11ab + 6cd$.
4. From $ab + cd - ac - bd$ take $ab + cd + ac + bd$.
5. From $m^2 + 3n^3$ subtract $-4m^2 - 6n^3 + 71x$.
6. $7xy - (-3xy) = ?$
7. $-9x^2y - (+6x^2y) - (-20x^2y) = ?$
8. $32x^2 - (-12y^2) - (+14x^2) - (+9y^2) + (-2y^2) = ?$
9. $28a^2b^2 - (+17a^2b^2) - (-19x^2y) - (+15x^2y) - (-5a^2b^2) = ?$

From

10. $-8x^2y^2 + 15x^3y + 13xy^3$ take $4x^2y^2 + 7x^3y - 8xy^3$.
11. $a^2bc + b^2ca + c^2ab$ take $3a^2bc - 5b^2ca - 4c^2ab$.
12. $-7a^2b + 8ab^2 + cd$ take $5a^2b - 7ab^2 + 6cd$.
13. $10a^2b^2 + 15ab^2 + 8a^2b$ take $-10a^2b^2 + 15ab^2 - 8a^2b$.
14. $b^3 + c^3 - 2abc$ take $a^3 + b^3 - 3abc$.
15. $7abc - 3a^3 + 5b^3 - c^3$ take $a^3 + b^3 + c^3 - 3abc$.
16. $\frac{1}{2}x^2 - \frac{1}{3}xy - \frac{3}{2}y^2$ take $-\frac{3}{2}x^2 + xy - y^2$.

17. $\frac{3}{8}x^2 - \frac{2}{3}ax$ take $\frac{1}{3} - \frac{1}{4}x^2 - \frac{5}{6}ax$.

18. $\frac{1}{8}a^3 - 2ax^2 - \frac{1}{3}a^2x$ take $\frac{1}{3}a^2x + \frac{1}{4}a^3 - \frac{3}{2}ax^2$.

If $A = a^2 - 4ab - 3b^2$, $\quad B = ab - 4b^2 - 3a^2$,

$C = b^2 - 4a^2 - 3ab$, $\quad D = 2a^2 + 2b^2 + 2c^2$,

find the expression for

19. $A + B + C + D$.

20. $A + B + C - D$.

21. $A + B - C - D$.

22. $A - B - C - D$.

23. $-A - B + C + D$.

24. $-A + B - C + D$.

In solving example 20, under the values of A, B, and C write that of D with its quality changed, and then add the results.

25. From $5x^3 + 3x - 1$ take the sum of $2x - 5 + 7x^2$ and $3x^2 + 4 - 2x^3 + x$.

26. From the sum of $2a^3 - 3a^2 + a - 2$ and $2 + 8a^2 - a^3$ subtract $3a - 7a^3 + 5a^2$.

27. From the sum of $4x^3 + 3x - 7$, $2x^2 - 3x + 2x^3 - 1$, and $-5x^3 + 2x - x^2 + 9$ take the sum of $2x^3 - 11x$ and $9x^3 + 5x^2 + 3 - 2x$.

68. Removal of signs of grouping. The *converse* of the associative law for addition in § 36 is

$$a + (b + c) \equiv a + b + c. \tag{1}$$

That is, *a sign of grouping preceded by the sign + can be removed if each enclosed term is left unchanged.*

Observe that the sign + is understood before b within the parentheses.

Ex. 1. $a + (4a - 7y + 5z) \equiv a + 4x - 7y + 5z$.

Ex. 2. $z + (-3x + 2y - 4a) \equiv z - 3x + 2y - 4a$.

By the rule for subtraction in § 67, we have

A sign of grouping preceded by the sign — can be removed, if the sign before each enclosed term is changed from + to —, or from — to +.

Ex. 1. $5a-(3b-2a+4c) \equiv 5a-3b+2a-4c$ (1)

$$\equiv 7a-3b-4c.$$

The sign + is understood before $3b$ within the parentheses.

Ex. 2. $-(5m-4n)-(-3m+7n) \equiv -5m+4n+3m-7n$ (2)

$$\equiv -2m-3n.$$

69. Sometimes one sign of grouping is enclosed within another; in this case the different signs of grouping must be of different *shapes* to avoid confusion.

When there are several signs of grouping they can be removed *one at a time* by the rules of § 68; and it is better for beginners to remove at every stage the *innermost* sign of grouping.

Ex. Removing the signs of grouping, simplify the expression

$$a-[x+\{y-(b-c)\}-z].$$

$$\begin{aligned} a-[x+\{y-(b-c)\}-z] &\equiv a-[x+\{y-b+c\}-z] \\ &\equiv a-[x+y-b+c-z] \\ &\equiv a-x-y+b-c+z. \qquad (1) \end{aligned}$$

In the above process the parentheses ( ) were removed first, then the braces { }, and then the brackets [ ].

Verify (1) when $a=8,\ x=3,\ y=-2,\ b=-3,\ c=-4,\ z=7.$

Removing the outer sign of grouping first, we have

$$\begin{aligned} a-[x+\{y-(b-c)\}-z] &\equiv a-x-\{y-(b-c)\}+z \\ &\equiv a-x-y+(b-c)+z \\ &\equiv a-x-y+b-c+z. \end{aligned}$$

In review, the student should begin with the outer sign of grouping, as he can thereby soon learn to remove, without error, two or more signs of grouping at a time.

Exercise 23.

Simplify each of the following expressions by removing the signs of grouping and combining like terms:

1. $a-(b+c)+(b-c-a)$.
2. $3x-(y-2x)+(z+y-5x)$.
3. $z-\{y-(z-x)\}$.
4. $3x-\{2y+5z-(3x+y)\}$.
5 $a-[a-\{a-(2a-a)\}]$.

Verify the results of examples 1 to 5 inclusive,

6. When $a=7,\ b=-3,\ c=4,\ x=10, y=-5,\ z=-2$.
7. When $a=-5,\ b=2,\ c=-1,\ x=-3,\ y=4,\ z=-7$.
8. $a+b-[a-b+\{a+b-(a-b)\}]$.
9. $x-(y-z)+\{2z-\overline{3y-5x}\}$.
10. $2a-\{3b+(4c-\overline{3b+2a})\}$.
11. $a-2b-\{3a-(b-c)-5c\}$.
12. $a-[3b+\{3c-(d-b)+a\}-2a]$.
13. $2x-(5y-\overline{3z+7})-[4+\{x-(3y+2z+5)\}]$.
14. $3a-[2b-\{4c-12a-(4b-8c)\}-(6b-12c)]$.
15. $-[15x-\{14y-(15z+12y)-(10x-15z)\}]$.
16. $-[a-\{a+(x-a)-(x-a)-a\}-2a]$.
17. $2x-(3y-4z)-\{2x-(3y+4z)\}-\{3y-(4z+2x)\}$.

70. Insertion of signs of grouping. Law (B) in § 36 is

$$a+b+c\equiv a+(b+c).$$

That is, *any number of terms of a polynomial can be enclosed within a sign of grouping preceded by the sign +, if each enclosed term is left unchanged.*

Ex. 1. $5x - 7y + 4c - 7b \equiv 5x + (-7y + 4c - 7b)$.

Ex. 2. $4a + 3c - 5x - 3y \equiv 4a + (+3c - 5x - 3y)$.

From the rule for subtraction it follows that,

Any number of terms of a polynomial can be enclosed within a sign of grouping preceded by the sign $-$, if the sign before each enclosed term is changed from $+$ to $-$ or from $-$ to $+$.

Ex. 1. $7x + 6y - 5a + 7c \equiv 7x - (-6y + 5a - 7c)$.

Ex. 2. $ax^3 - 2cx - cx^3 + bx^2 - x \equiv ax^3 - cx^3 + bx^2 - 2cx - x$

$$\equiv (a - c)x^3 + bx^2 - (2cx + x)$$

$$\equiv (a - c)x^3 + bx^2 - (2c + 1)x.$$

Exercise 24.

In each of the following expressions enclose the last four terms within a sign of grouping preceded by the sign $-$, without changing the value of the expression:

1. $3x - 2a + 5b - y + z$.

2. $a - b - x + 3b - z + 2y$.

3. $3y + 2x + 7z + a + 2b + c$.

4. $2z - 7x - 2a - 3b - 5c - 9y$.

Simplify each of the following expressions by combining the terms having the same powers of x, so as to have the sign $+$ before each sign of grouping:

5. $ax^4 + bx^2 + 5 + 2bx - 5x^2 + 2x^4 - 3x$.

   Ans. $(a + 2)x^4 + (b - 5)x^2 + (2b - 3)x + 5$.

6. $3bx^2 - 7 - 2x + ab + 5ax^3 + cx - 4x^2 - bx^3$.

7. $2 - 7x^3 + 5ax^2 - 2cx + 9ax^3 + 7x - 3x^2$.

8. $2cx^5 - 3abx + 4dx - 3bx^4 - a^2x^5 + x^4$.

Simplify each of the following expressions by combining like terms in x so as to have the sign $-$ before each sign of grouping:

9. $ax^2 + 5x^3 - a^2x^4 - 2bx^3 - 3x^2 - bx^4.$

10. $7x^3 - 3c^2x - abx^5 + 5ax + 7x^5 - abcx^3.$

11. $ax^2 + a^2x^3 - bx^2 - 5x^2 - cx^3.$

12. $3b^2x^4 - bx - ax^4 - cx^4 - 5c^2x - 7x^4.$

Simplify the following expressions, and in each result add the terms involving like powers of x:

13. $ax^3 - 2cx - [bx^2 - \{cx - dx - (bx^3 + 3cx^2)\} - (cx^2 - bx)].$

14. $5ax^3 - (7bx - 7cx^2) - \{6bx^2 - (3ax^2 + 2ax) - 4cx^3\}.$

Express in descending powers of x the sum of,

15. $a^2x^3 - 5x,\ 2ax^2 - 5ax^3,\ 2x^3 - bx^2 - ax.$

16. $ax^2 + bx - c,\ qx - r - px^2,\ x^2 + 2x + 3.$

17. $px^3 - qx,\ qx^2 - px,\ q - x^3,\ px^2 + qx^3.$

18. $2ax^3 - 3cx^2 + px,\ 3nx - mx^3 - 2cx^2,\ x - 2x^2 - 3x^3.$

19. $bx - ax^2 - bx^3,\ 3x^3 - 4nx - 2mx^2,\ 2x^2 - px^3.$

20. $cx^2 - 2ax + mx^3,\ 4x^2 - bx^3,\ 4nx + 2px^3,\ 3x^3 - 2rx^2 - x.$

CHAPTER V

MULTIPLICATION OF INTEGRAL LITERAL EXPRESSIONS

71. The **degree of an integral term** is the number of its *literal* factors. But we usually speak of the degree of a term in regard to one or more of its letters.

E.g., $5\,ax$ is of the *second* degree, and $7\,a^2x^3$ is of the *fifth* degree. Again $4\,abx^2y^3$, which is of the *seventh* degree, is of the *first* degree *in* a, of the *second* degree *in* x, of the *third* degree *in* y, and of the *fifth* degree *in* x and y.

72. The **degree of a polynomial** is the degree of its term of highest degree.

E.g., the trinomial $ax^2 + bx + c$ is of the *first* degree *in* a, b, or c, and of the *second* degree *in* x. The binomial $ax^2y + by^2$, which is of the *fourth* degree, is of the *second* degree *in* x or y, and of the *third* degree *in* x and y. The trinomial $ax^2 + 2\,bxy + cy^2$ is of the *second* degree *in* x, *in* y, and *in* x and y.

73. An expression is said to be **homogeneous** in one or more letters when all its terms are of the same degree in these letters.

E.g., $2\,a^2 + 3\,ab + 4\,b^2$ is homogeneous in a and b;

$5\,x^3 + 3\,x^2y + 8\,xy^2 + y^3$ is homogeneous in x and y;

and $ax^2 + 2\,bxy + cy^2$ is homogeneous in x and y.

Exercise 25.

What is the degree of the term $3\,a^2bx^3y^4$,

1. In a? 3. In x? 5. In a and b? 7. In a, x, and y?

2. In b? 4. In y? 6. In x and y? 8. In b, x, and y?

What is the degree of the trinomial

$$a^2x^4 + 7\,a^3b^2x^3y^2 - 5\,abxy^2,$$

9. In x? **10.** In a? **11.** In x and y? **12.** In b and y?

Write two trinomials of the third degree and homogeneous,

13. In a and b. **14.** In x and y. **15.** In a and x.

74. *A product is zero when one of its factors is zero.*

That is, $\quad \boldsymbol{a \cdot 0 \equiv 0}$ and $\boldsymbol{0 \cdot a \equiv 0}$.

Proof. $\quad a \cdot 0 \equiv a(b-b),$ §§ 11, 32

$\equiv ab - ab \equiv 0.$ §§ 60, 11

Similarly, $\quad 0 \cdot a \equiv (b-b)\,a \equiv 0.$

Conversely, *when a product is* 0, *one or more of its factors is* 0.

That is, if $a \cdot b = 0$, then $a = 0$, or $b = 0$, or $a = 0$ and $b = 0$.

75. *Any positive integral power of* 0 *is* 0 ; *that is* $0^n \equiv 0$.

Proof. $\quad 0^n \equiv 0 \cdot 0 \cdot 0 \cdots$ to n factors $\equiv 0.$ § 74

76. Product of powers of same base.

Ex. 1. $2^3 \times 2^2 = (2 \times 2 \times 2)(2 \times 2) = 2 \times 2 \times 2 \times 2 \times 2 = 2^5.$

Ex. 2. $a^3a^2 \equiv (aaa)(aa) \equiv aaaaa \equiv a^5.$

These examples illustrate the following **law of exponents.**

The product of the **m***th power and the* **n***th power of the same base is equal to the* $(\boldsymbol{m}+\boldsymbol{n})$*th power of that base, and conversely.*

That is, $\quad \boldsymbol{a^m \cdot a^n \equiv a^{m+n}}.$

Proof. $a^ma^n \equiv (aaa \cdots \text{to } m \text{ factors})(aaa \cdots \text{to } n \text{ factors})$ § 52

$\equiv aaa \cdots$ to $m+n$ factors § 49

$\equiv a^{m+n}.$ § 52

Ex. Multiply $3\,a^2x^3$ by $-4\,a^4x^2y$.

$$\begin{aligned}(3\,a^2x^3)(-4\,a^4x^2y) &\equiv 3\,a^2x^3\,(-4)\,a^4x^2y && \S\,49\\ &\equiv 3\,(-4)\cdot a^2a^4\cdot x^3x^2\cdot y && \S\,49\\ &\equiv -12\,a^6x^5y. && \S\S\,47,\,76\end{aligned}$$

This example illustrates the method of finding the

77. Product of two or more monomials. Using the commutative and associative laws, we have the following rule:

Multiply together their numeral factors, observing the law of quality; after this write the product of their literal factors, observing the law of exponents.

Ex. Multiply together $-5\,ay^2$, $-2\,a^2x^3$, and $-9\,ax^2y$.

$$\begin{aligned}(-5\,ay^2)(-2\,a^2x^3)(-9\,ax^2y) &\equiv (-5)(-2)(-9)\,aa^2ax^3x^2y^2y\\ &\equiv -90\,a^4x^5y^3.\end{aligned}$$

Exercise 26.

Find the product of:

1. a^3 and a^4.
2. a^7 and a^5.
3. y^2, y^3, and y^4.
4. ax and $3\,ax$.
5. $-2\,abx$ and $-7\,ab$.
6. $6\,x^2y$ and $-10\,axy$.
7. $-3\,a^2b$ and $12\,ab^3$.
8. $-abcd$ and $-3\,a^2b^2c$.
9. $7\,x^2y^2z^3$ and $-5\,x^2y^3z$.
10. $-3\,a^2b^2c^2$ and $8\,a^3b^5c^4d$.
11. $2\,ab$, $-4\,a^2b$, and $5\,ab^3$.
12. $-5\,ax$, $-7\,a^2x$, and $2\,ax^3$.
13. $8\,xy^3$, $-3\,x^2y$, and $-3\,xy$.
14. $-7\,ab^2$, $-3\,a^2b^3$, and $-a^2b^2$.
15. a^2b^2c, $2\,ab^3c$, and $-5\,abc$.
16. $-7\,x^2y^3$, x^3y^4, and axy.
17. $-a^2bx$, ab^2x, and $-ax^5$.
18. $-a^2x^3$, $-b^2x$, and $-aby$.

78. Multiplication of a polynomial by a monomial. The distributive law of multiplication is

$$(a+b+c+\cdots)x \equiv ax+bx+cx+\cdots.$$

That is, to multiply a polynomial by a monomial, *multiply each term of the polynomial by the monomial, and add the several products.*

Ex. Multiply $2z^2-4y^3$ by $-3yz$.

$$(2z^2-4y^3)(-3yz)=(2z^2)(-3yz)+(-4y^3)(-3yz)$$
$$=-6yz^3+12y^4z.$$

Writing the multiplier under the multiplicand, the work can be arranged as at the right of the page.

$$\begin{array}{r} 2z^2-4y^3 \\ -3yz \\ \hline -6yz^3+12y^4z \end{array}$$

Exercise 27.

Multiply:

1. $4a^2-5a+3b$ by $2a^3$.
2. $2a^2+3ab+2b^2$ by $-3a^2b^2$.
3. $bc+ca-ab$ by abc.
4. $2x^3-3x^2+5x-4$ by $-5x^2$.
5. $-4x^4+3x^3-3x^2+4$ by $-6x^3$.
6. $9gh-12ga-3gb$ by $3gh$.
7. $-a^2bc+b^2ca-c^2ab$ by $-ab$.
8. $-5xy^2z+3xyz^2-8x^2yz-7xyz$ by $-2xyz$.
9. $a^2b^2c^2-abc-ax-by-cz$ by $-5abcxy$.
10. $\frac{6}{7}a^2x^2-\frac{3}{2}ax^3+\frac{1}{5}ax$ by $-\frac{7}{3}a^3x$.
11. $-\frac{3}{2}xy^2+\frac{1}{2}axy-\frac{2}{5}ay^2+\frac{4}{3}a^2x$ by $-\frac{2}{3}axy$.
12. $\frac{7}{11}x^2y^3-\frac{2}{3}xy^2+\frac{1}{2}axy-\frac{1}{5}x^3y^2$ by $\frac{11}{5}x^2y$.
13. $(x+y)^2-2a(x+y)+5a^3$ by $2(x+y)$.
14. $(x+1)^3-4a(x+1)^2-2ab$ by $-5ab(x+1)^2$.
15. $(a^2+1)^3-3x(a^2+1)^2-4xy$ by $-3x^2y(a^2+1)^4$.
16. $(x^2+y)^4-a(x^2+y)^2+3a^2b^3$ by $-4a^3b^4(x^2+y)^3$.

Remove the signs of grouping, and simplify each of the following expressions:

17. $(a+b)c-(a-b)c.$

18. $2(a-b)+4(a+b).$

19. $\frac{1}{2}(b-2c)+\frac{3}{4}(c-2b).$

20. $7a(b-c)-2b(a-c).$

21. $a^2b^2(c^2-d^2)+c^2d^2(a^2-b^2)+b^2c^2(d^2-a^2).$

22. $2\{3ab-4a(c-2b)\}.$

23. $7ac-2\{2c(a-3b)-3(5c-2b)a\}.$

79. **To multiply one polynomial by another,**

Multiply each term of the multiplicand by each term of the multiplier, and add the resulting products.

Proof. Let $x+y+z$ be the multiplicand, and $a+b$ the multiplier; then by successive applications of the distributive law, we have

$$\begin{aligned}(x+y+z)(a+b) &\equiv x(a+b)+y(a+b)+z(a+b)\\ &\equiv xa+ya+za+xb+yb+zb. \qquad \S\,36\end{aligned}$$

Similarly when each factor has any number of terms.

Ex. 1. Multiply $-2x+3y$ by $4x-7y$.

$$\begin{aligned}&(-2x+3y)(4x-7y)\\ &\quad\equiv(-2x)\cdot 4x+3y\cdot 4x+(-2x)(-7y)+3y(-7y) &(1)\\ &\quad\equiv -8x^2+12xy+14xy-21y^2 &(2)\\ &\quad\equiv -8x^2+26xy-21y^2. &(3)\end{aligned}$$

Performing the steps in (1) and (2) mentally, we can arrange the work as below:

$$\begin{array}{r}-2x+3y\\ 4x-7y\\ \hline -8x^2+26xy-21y^2\end{array}$$

Observe that the first and last terms in the product are the products of terms in the vertical lines, while the second term is the sum of the products of the terms in the diagonal lines.

In this way solve the first 15 examples in Exercise 28.

Ex. 2. Multiply $x^2 - 2x + x^3 + 1$ by $2 - x + x^2$.

Arranging both multiplicand and multiplier in *descending* powers of x, we have

$$
\begin{array}{rrrrrrl}
 & x^3 & + x^2 & - 2x & + 1 & & (1) \\
 & x^2 & - x & + 2 & & & \\
\hline
x^5 & + x^4 & - 2x^3 & + x^2 & & & (2) \\
 & - x^4 & - x^3 & + 2x^2 & - x & & (3) \\
 & & 2x^3 & + 2x^2 & - 4x & + 2 & (4) \\
\hline
x^5 & & - x^3 & + 5x^2 & - 5x & + 2 &
\end{array}
$$

Expression (2) is the product of (1) multiplied by x^2;
Expression (3) is the product of (1) multiplied by $-x$; and
Expression (4) is the product of (1) multiplied by 2.
The sum of these partial products is the required product, by § 79.

A vertical line, or *bar*, is often a convenient *sign of grouping*. Its use is illustrated in the next example.

Ex. 3. Multiply $x^3 - 2x^2y + 3xy^2 - y^3$ by $x^2 - 3xy + y^2$.

$$
\begin{array}{l}
x^3 - 2x^2y + 3xy^2 - y^3 \\
x^2 - 3xy + y^2 \\
\hline
\end{array}
$$

$$
\begin{array}{rr|rr|rr|rr|r}
x^5 & -2 & x^4y & +3 & x^3y^2 & -1 & x^2y^3 & & \\
 & -3 & & +6 & & -9 & & +3 & xy^4 \\
 & & & +1 & & -2 & & +3 & -y^5 \\
\end{array}
$$

$$x^5 - 5x^4y + 10x^3y^2 - 12x^2y^3 + 6xy^4 - y^5$$

The sum of the numbers before each bar is the coefficient of the literal factor after it.

In this example the multiplicand and the multiplier are both homogeneous. Observe that the product is homogeneous also.

This illustrates the following principle.

80. *The product of two or more homogeneous expressions is homogeneous.*

Proof. If the homogeneous multiplicand is of the nth degree, and the homogeneous multiplier is of the mth degree, then each term in the product will be of the $(m+n)$th degree; that is, the product will be a homo-

geneous expression of the $(m+n)$th degree; and so on for any number of factors.

When the multiplicand and the multiplier are homogeneous, that fact should be noted in every case by the pupil; and if the product obtained is not homogeneous, it is at once known that there is an error.

Exercise 28.

Multiply:

1. $x+2y$ by $x-2y$.
2. $2x+3y$ by $3x-2y$.
3. $a-3b$ by $a+3b$.
4. $x+7$ by $x-6$.
5. $3x-7$ by $2x-1$.
6. $2x-4$ by $2x+6$.
7. $2y+5b$ by $3y-4b$.
8. $2m^2+5n^2$ by $2m^2-n^2$.
9. $3m^2-1$ by $3m^2+1$.
10. $-x+7$ by $x-7$.
11. $-x-16$ by $-x+16$.
12. $-x+21$ by $x-21$.
13. $2a+\frac{1}{3}b$ by $3a+\frac{1}{2}b$.
14. $\frac{1}{4}a-\frac{1}{3}b$ by $\frac{1}{3}a-\frac{1}{4}b$.
15. $ax-by$ by $ax+by$.
16. x^2+x+1 by $x-1$.
17. a^2+ab+b^2 by $a-b$.
18. a^2-ab+b^2 by $a+b$.
19. $x^4-x^2y^2+y^4$ by x^2+y^2.
20. a^2-ab+b^2 by a^2+ab+b^2.
21. $a^2-2ax+4x^2$ by $a^2+2ax+4x^2$.
22. $10a^2+12ab+9b^2$ by $4a-3b$.
23. $a^2x-ax^2+x^3-a^3$ by $x+a$.
24. x^2+x-2 by x^2+x-6.
25. $2x^3-3x^2+2x$ by $2x^2+3x+2$.
26. $a^3+2a^2b+2ab^2$ by $a^2-2ab+2b^2$.
27. $x^2-3xy-y^2$ by $-x^2+xy+y^2$.
28. $x^2-2xy+y^2$ by $x^2+2xy+y^2$.
29. $27x^3-36ax^2+48a^2x-64a^3$ by $3x+4a$.
30. $ab+cd+ac+bd$ by $ab+cd-ac-bd$.

31. $x^{12} - x^9y^2 + x^6y^4 - x^3y^6 + y^8$ by $x^3 + y^2$.

32. $-2x^3y + y^4 + 3x^2y^2 + x^4 - 2xy^3$ by $x^2 + 2xy + y^2$.

33. $a^2 + b^2 + c^2 - bc - ca - ab$ by $a + b + c$.

34. $x^{n+2} + 2x^{n+1} - 3x^n - 1$ by $x + 1$.

35. $-ax^2 + 3axy^2 - 9ay^4$ by $-ax - 3ay^2$.

36. $-x^3y + y^4 + x^2y^2 + x^4 - xy^3$ by $x + y$.

37. $\frac{1}{2}a^2 + \frac{1}{3}a + \frac{1}{4}$ by $\frac{1}{2}a - \frac{1}{3}$.

38. $\frac{1}{2}x^2 - 2x + \frac{3}{2}$ by $\frac{1}{2}x + \frac{1}{3}$.

39. $\frac{2}{3}x^2 + xy + \frac{3}{2}y^2$ by $\frac{1}{3}x - \frac{1}{2}y$.

40. $\frac{1}{2}x^2 - \frac{2}{3}x - \frac{3}{4}$ by $\frac{1}{2}x^2 + \frac{2}{3}x - \frac{3}{4}$.

41. $\frac{3}{2}x^2 - ax - \frac{2}{3}a^2$ by $\frac{3}{4}x^2 - \frac{1}{2}ax + \frac{1}{3}a^2$.

42. $\frac{2}{3}ax + \frac{2}{3}x^2 + \frac{1}{3}a^2$ by $\frac{3}{4}a^2 + \frac{3}{2}x^2 - \frac{3}{2}ax$.

43. $3x^m - 2x^{m-1} + 4x^{m-2}$ by $2x^m + 3x^{m-1} - 4x^{m-2}$.

44. $3x^{n-3} + x^{n-2} - 2x^{n-1} - 4x^n$ by $2x^{n-3} + 3x^{n-4}$.

45. $4a^4x^{3n} - a^2x^{2n} + 5x^n$ by $a^2x^{2n-1} + 6x^{n-3}$.

46. $3a^{n-2}x^2 - a^{n-1}x^3 + a^n$ by $a^2x^{n-2} - 2x^{n-1} - 3ax^{n+1}$.

47. $4x^{2m+1} - 3x^{3m} - 2x^{m+1} + \frac{1}{3}x^{3m-1}$
by $\frac{1}{4}x^{3m+1} - 2x^{2m+1} - x^{3m-1}$.

48. $3(a+b)^3 - 2(a+b)^2(x-y) - 4(a+b)(x-y)^2 + 7(x-y)^3$
by $2(a+b)^2(x-y) - 6(a+b)(x-y)^2$.

81. Removal of signs of grouping.

Ex. Remove the signs of grouping, and simplify,

$$42 - 5[-12x - 3\{-15x + 3(8 - \overline{7 - 3x})\}].$$

$$\begin{aligned}\text{The expression} &= 42 - 5[-12x - 3\{-15x + 3(3x + 1)\}] \\ &= 42 - 5[-12x - 3\{-6x + 3\}] \\ &= 42 - 5[6x - 9] \\ &= 87 - 30x.\end{aligned}$$

Exercise 29.

Remove the signs of grouping, and simplify:

1. $3b-\{5a-[6a+2(10-b)]\}$.
2. $a-(b-c)-[a-b-c-2\{b+c\}]$.
3. $8(b+c)-[-\{a-b-3(c-b+a)\}]$.
4. $2(3b-5a)-7[a-6\{2-(5a-b)\}]$.
5. $6\{a-2[b-3(c+d)]\}-4\{a-3[b-4(c+d)]\}$.
6. $5\{a-2[a-2(a+x)]\}-4\{a-2[a-2(a+x)]\}$.
7. $-10\{a-6[a-(b-c)]\}+60\{b-(c+a)\}$.
8. $-3\{-2[-4(-a)]\}+5\{-2[-2(-a)]\}$.
9. $-2\{-1[-(x-y)]\}+\{-2[-(x-y)]\}$.

Multiply together the following expressions, and arrange each product in descending powers of x:

10. ax^2+bx+1 and $cx+2$.
11. $ax^2-2bx+3c$ and $x-1$.
12. x^3+ax^2-bx-c and x^3-ax^2-bx+c.
13. ax^3-x^2+3x-b and ax^3+x^2+3x+b.
14. $x^4-ax^3-bx^2+cx+d$ and $x^4+ax^3-bx^2-cx+d$.

82. Multiplication by detached coefficients.

The labor of multiplication is lessened by using the *method of detached coefficients* in the two following cases:

(i) When two polynomial factors contain but one letter.

Ex. 1. Multiply $4x^3-3x^2+2x-5$ by $5x^2+3x-4$.

Writing coefficients only, we proceed as below:

$$\begin{array}{rrrrrr}
4 & -\ 3 & +\ 2 & -\ 5 & & \\
5 & +\ 3 & -\ 4 & & & \\
\hline
20 & -\ 15 & +\ 10 & -\ 25 & & \\
 & +\ 12 & -\ 9 & +\ 6 & -\ 15 & \\
 & & -\ 16 & +\ 12 & -\ 8 & +\ 20 \\
\hline
20 & -\ 3 & -\ 15 & -\ 7 & -\ 23 & +\ 20
\end{array}$$

Inserting the literal factors, whose law of formation is seen by inspection, we have for the complete product,

$$20x^5 - 3x^4 - 15x^3 - 7x^2 - 23x + 20.$$

(ii) When each of two polynomial factors is homogeneous and contains only two letters.

Ex. 2. Multiply $5a^4 + 4a^3b - 3ab^3 + 2b^4$ by $a^2 - 2b^2$.

$$\begin{array}{l}
5 + 4 + 0 - 3 + 2 \\
1 + 0 - 2 \\
\hline
5 + 4 + 0 - 3 + 2 \\
\qquad\;\; - 10 - 8 - 0 + 6 - 4 \\
\hline
5 + 4 - 10 - 11 + 2 + 6 - 4
\end{array}$$

In the first expression, the term containing a^2b^2 is lacking; that is, its coefficient is zero, which is written in the line of coefficients. In the second expression, the term containing ab is missing; hence its coefficient is zero.

In the method of detached coefficients, the zero coefficients must evidently be written with the other coefficients.

Inserting the literal factors, whose law of formation is seen by inspection, we have for the complete product,

$$5a^6 + 4a^5b - 10a^4b^2 - 11a^3b^3 + 2a^2b^4 + 6ab^5 - 4b^6.$$

Observe that the entire number of coefficients (zero coefficients being included) in the product is one less than the number of coefficients in both the multiplicand and multiplier together.

Exercise 30.

1. Multiply $x^5 + 2x^4 - x^2 + 3x - 1$ by $x^3 - 2x - 3$.
2. Multiply $3a^4 + 2a^2 - 5a + 4$ by $2a^3 - 3a - 2$.
3. Multiply $x^3 + 5x^2y - 4xy^2 + 3y^3$ by $2x^3 - 3x^2y + y^3$.
4. Multiply $3a^5 - 2a^4b - 4a^3b^2 - ab^4$ by $a^2 - 2b^2$.
5. Multiply $4x^4 - 3x^3y + 7xy^3 + 2y^4$ by $x^3 + 3y^3$.
6. Rework by detached coefficients those examples in exercise 28, from 19 to 42, to which the method is applicable.

CHAPTER VI

DIVISION OF INTEGRAL LITERAL EXPRESSIONS

83. Division is the *inverse* of *multiplication.* Having given a product and one factor, *division* is the operation of finding the other factor.

That is, *to divide* one number by another is to find a third number which multiplied by the second number gives the first.

Thus, $-12 \div 3 = -4$; for $-4 \times 3 = -12$,

and $-12 \div (-3) = 4$; for $4 \times (-3) = -12$.

As in Arithmetic, the given product is called the **dividend**, the given factor the **divisor**, and the required factor the **quotient**.

84. Law of Quality. In each of the following identities the third number multiplied by the second gives the first; hence by definition the third number in each case is the quotient of the first divided by the second.

$$\left.\begin{array}{ll} {}^{+}(ab) \div {}^{+}a \equiv {}^{+}b; & {}^{-}(ab) \div {}^{-}a \equiv {}^{+}b; \\ {}^{+}(ab) \div {}^{-}a \equiv {}^{-}b; & {}^{-}(ab) \div {}^{+}a \equiv {}^{-}b. \end{array}\right\} \qquad (1)$$

From identities (1) it follows that,

The quotient is positive when the dividend and the divisor are like in quality; and negative when they are opposite in quality.

The arithmetic value of the quotient is equal to the quotient of the arithmetic value of the dividend by that of the divisor.

Any number divided by $^{+}1$ is equal to the number itself.

Any number divided by $^{-}1$ is equal to its arithmetically equal opposite number.

Exercise 31.

Perform each of the following indicated operations:

1. $-25 \div 5$.
2. $36 \div (-6)$.
3. $-51 \div (-3)$.
4. $-33 \div (-1)$.
5. $75 \div (-25)$.
6. $-72 \div (-6)$.
7. $-105 \div (-21)$.
8. $-144 \div 24$.
9. $21 \div (-1)$.
10. $-36 \div 4$.
11. $-1 \div \frac{2}{3}$.
12. $1 \div (-\frac{3}{4})$.

Find the value of $(x + y) \div z$,

13. When $x = -15$, $y = -3$, $z = 6$.
14. When $x = -48$, $y = 6$, $z = -$

Find the value of $(x - y) \div (a + b)$,

15. When $x = 22$, $y = -2$, $a = 5$, $b = 3$.
16. When $x = -21$, $y = 6$, $a = -7$, $b = 6$.

85. From the definition of division we have

quotient $\times$ divisor $\equiv$ dividend.

That is, since the quotient of N divided by a is $N \div a$, we have,

$$(N \div a) \times a \equiv N. \tag{1}$$

86. The **reciprocal** of a number is 1 divided by that number.

Since their product is $+1$, any number and its reciprocal have the same quality.

E.g., the reciprocal of 4 is $\frac{1}{4}$; the reciprocal of -4 is $1 \div (-4)$ or $-\frac{1}{4}$; and the reciprocal of $-\frac{2}{3}$ is $1 \div (-\frac{2}{3})$, or $-\frac{3}{2}$.

87. *Dividing by any number except zero gives the same result as multiplying by the reciprocal of that number.*

That is, $$N \div a \equiv N \times (1 \div a). \qquad (1)$$

Proof. The second member of (1) multiplied by a is, by § 85, equal to N; hence it is the quotient of N divided by a.

Ex. 1. $16 \div 4 = 16 \times \frac{1}{4} = 4.$

Ex. 2. $16 \div (-4) = 16 \times (-\frac{1}{4}) = -4.$

88. The commutative law for division.

Ex. 1. $-40 \div (-2) \div (-5) = -40 \times (-\frac{1}{2}) \times (-\frac{1}{5}) = -4. \quad (1)$

Ex. 2. $\frac{2}{5} \div (-\frac{2}{3}) \div (-\frac{3}{4}) = \frac{2}{5} \times (-\frac{3}{2}) \times (-\frac{4}{3}) = \frac{4}{5}. \quad (2)$

Since we can change the order of the factors in the second member of either (1) or (2), we can also change the order of the divisors in the first member of either identity; this illustrates that,

The **commutative law** holds for division as well as for multiplication, provided *the sign of operation,* $\div$ *or* $\times$, *before each number is transferred with the number itself.*

That is, $$\boldsymbol{N \times b \div c \equiv N \div c \times b}. \qquad (1)$$

Proof.
$$\begin{aligned} N \times b \div c &\equiv N \times b \times (1 \div c) && \S\ 87 \\ &\equiv N \times (1 \div c) \times b && \S\ 49 \\ &\equiv N \div c \times b. && \S\ 87 \end{aligned}$$

Ex. $(-60) \times (-22) \div (-15) = (-60) \div (-15) \times (-22)$
$= 4 \times (-22) = -88.$

89. *A product of two or more factors is divided by a number if any one of the factors is divided by that number.*

Proof. $(ab) \div c \equiv a \div c \times b \equiv (a \div c)b, \qquad \S\ 88$

or $(ab) \div c \equiv b \div c \times a \equiv (b \div c)a. \qquad \S\ 88$

90. Any indicated quotient is called a **fraction.**

A quotient is often indicated by placing the dividend over the divisor with a line between them.

E.g., $a \div b$, $\frac{a}{b}$, and a/b are but different ways of indicating that a is to be divided by b.

Each of these expressions is a fraction, a being the *dividend*, and b the *divisor.* The dividend and divisor of a fraction are often called its *numerator* and *denominator* respectively.

When the dividend or divisor consists of more than one term, the horizontal dividing line in a fraction serves as a sign both of *division* and of *grouping.*

E.g., in the fraction $\frac{a-b}{c+d}$ the horizontal dividing line takes the place of both the sign of division and the two parentheses in the form $(a-b)\div(c+d)$, or $(a-b)/(c+d)$.

In § 1 any fractional number as $5/6$ was regarded as $(1/6)\times 5$; but it can also be regarded as $5 \div 6$; for

$$N \div a \equiv N \times (1 \div a) \equiv (1/a) \times N. \qquad \text{§§ 87, 49}$$

91. The **product of two or more fractions** *is equal to the product of their dividends divided by the product of their divisors; and conversely.*

That is,
$$\frac{a}{x}\cdot\frac{b}{y}\cdot\frac{c}{z}\equiv\frac{abc}{xyz}. \qquad (1)$$

Proof.
$$\frac{a}{x}\cdot\frac{b}{y}\cdot\frac{c}{z}\cdot x\cdot y\cdot z\equiv\frac{a}{x}\cdot x\cdot\frac{b}{y}\cdot y\cdot\frac{c}{z}\cdot z \qquad \text{§ 49}$$
$$\equiv\left(\frac{a}{x}\cdot x\right)\left(\frac{b}{y}\cdot y\right)\left(\frac{c}{z}\cdot z\right) \qquad \text{§ 49}$$
$$\equiv abc. \qquad \text{§ 85}$$

Dividing each member by xyz, we obtain (1).

Ex. $\frac{4}{-5}\times\frac{3}{-7}\times\frac{-2}{-3}=\frac{-(4\times 3\times 2)}{-(5\times 7\times 3)}=\frac{8}{35}.$

92. Quotient of powers of the same base.

Ex. 1. $a^5 \div a^2 \equiv a^{5-2} \equiv a^3$; for $a^3 \times a^2 \equiv a^5$.

Ex. 2. $a^7 \div a^3 \equiv a^{7-3} \equiv a^4$; for $a^4 \times a^3 \equiv a^7$.

These examples illustrate the following law:

If $m > n$, *the quotient of the* m*th power of any base divided by the* n*th power of the same base is equal to the* $(m-n)$*th power of that base; and conversely.*

That is, $$a^m \div a^n \equiv a^{m-n}.$$

Proof. $$a^{m-n} \times a^n \equiv a^{m-n+n} \equiv a^m. \qquad \text{§ 76, 83}$$

Ex. 1. Divide $20\,a^4b^5$ by $-5\,ab^3$.

$$\frac{20\,a^4b^5}{-5\,ab^3} \equiv \frac{20}{-5} \cdot \frac{a^4}{a} \cdot \frac{b^5}{b^3} \qquad \text{§ 91}$$
$$\equiv -4\,a^3b^2. \qquad \text{§§ 84, 92}$$

Ex. 2. Divide $-5\,a^2b^4x^3$ by $11\,a^2b^2x^2$.

$$\frac{-5\,a^2b^4x^3}{11\,a^2b^2x^2} \equiv \frac{-5}{11} \cdot \frac{a^2}{a^2} \cdot \frac{b^4}{b^2} \cdot \frac{x^3}{x^2} \qquad \text{§ 91}$$
$$\equiv -\tfrac{5}{11} \cdot 1 \cdot b^2 \cdot x \equiv -\tfrac{5}{11}\,b^2x.$$

These examples illustrate the following section.

93. The quotient of one monomial by another. By the converse of § 91 we have the following rule:

Divide the numeral factor of the dividend by that of the divisor, observing the law of quality; after this write the quotient of their literal factors, observing the law of exponents.

Ex. 1. $-84\,a^5x^3 \div 12\,a^4x \equiv -7\,ax^2$.

Ex. 2. $77\,a^2x^3y^4 \div (-7\,ax^2y) \equiv -11\,axy^3$.

Check. Multiplying the obtained quotient by the given divisor, we obtain the dividend; hence, the division is correct.

Exercise 32.

Divide:

1. $-72a^2$ by $-9a$.
2. $84a^3$ by $-7a^2$.
3. $-35x^3$ by $7x^3$.
4. $4a^2b^3c^3$ by $-ab^2c^2$.
5. $-12a^6b^6c^6$ by $-3a^4bc^2$.
6. $84x^2y^6z^5$ by $-7xy^2z^3$.
7. $28a^4b^3$ by $-4a^3b$.
8. $-35a^2b^3$ by $5ab$.
9. $-16x^2y^6$ by $-4xy^4$.
10. $36m^{10}a^{12}$ by $9m^6a^9$.
11. $96a^4x^8z^4$ by $12a^3x^5z^2$.
12. $-256x^6y^9z^{11}$ by $-8x^4y^5z^6$.
13. $84a^2b^7c^5$ by $14ab^2c^3$.
14. $-16b^2yx^2$ by $-2xy$.
15. $50y^3x^3$ by $-5x^3y$.
16. $-144a^7z^9$ by $-24a^4z^8$.
17. $-3x^{m+2}$ by $5x^{m+1}$.
18. $-4x^{m+n}y^{m+n}$ by $7x^my^m$.
19. $5x^{n+3}y^{n+5}$ by $-8x^ny^n$.
20. $-7x^{n+1}y^{m+2}$ by $-2x^{n-1}y^{m-2}$.
21. $-42x^{n+3}a^{m-1}$ by $-7x^{n-2}a^{m-3}$.
22. $-50x^{n+a}y^{m+b}$ by $25x^{n-b}y^{m-a}$.

94. **Distributive law for division.** *The quotient of one expression divided by another is equal to the sum of the results obtained by dividing the parts of the first expression by the second; and conversely.*

That is, $$\frac{a+b+c+\cdots}{x} \equiv \frac{a}{x}+\frac{b}{x}+\frac{c}{x}+\cdots. \qquad (C')$$

Principle (C') lies at the basis of division in Arithmetic; *e.g.*, to divide 894 by 6 we separate 894 into the parts 600, 240, and 54, divide each of these parts by 6, and add the results.

Thus $$\frac{894}{6}=\frac{600}{6}+\frac{240}{6}+\frac{54}{6}=100+40+9=149.$$

Proof.
$$\frac{a+b+c+\cdots}{x} \equiv (a+b+c+\cdots)\frac{1}{x} \qquad \S\ 87$$
$$\equiv a\frac{1}{x}+b\frac{1}{x}+c\frac{1}{x}+\cdots \qquad \S\ 60$$
$$\equiv \frac{a}{x}+\frac{b}{x}+\frac{c}{x}+\cdots. \qquad \S\ 87$$

95. To divide a polynomial by a monomial.

By the distributive law for division, in § 94, we have the following rule:

Divide each term of the polynomial by the monomial, and add the resulting quotients.

Ex. 1. Divide $12\,x^3 - 5\,ax^2 - 2\,a^2x$ by $3\,x$.

$$\frac{12\,x^3-5\,ax^2-2\,a^2x}{3\,x} \equiv \frac{12\,x^3}{3\,x}+\frac{-5\,ax^2}{3\,x}+\frac{-2\,a^2x}{3\,x} \qquad \S\ 94$$
$$\equiv 4\,x^2-\tfrac{5}{3}\,ax-\tfrac{2}{3}\,a^2.$$

Ex. 2. Divide $12\,a^3 + 9\,a^4 - 6\,a^5$ by $-3\,a^2$.

$$\frac{12\,a^3+9\,a^4-6\,a^5}{-3\,a^2} \equiv \frac{12\,a^3}{-3\,a^2}+\frac{9\,a^4}{-3\,a^2}+\frac{-6\,a^5}{-3\,a^2} \qquad \S\ 94$$
$$\equiv -4\,a-3\,a^2+2\,a^3.$$

Exercise 33.

Divide:

1. $5\,x^2 - 7\,ax + 4\,x$ by x.
2. $x^6 - 7\,x^5 + 4\,x^4$ by x^2.
3. $10\,x^7 - 8\,x^6 + 3\,x^4$ by x^3.
4. $27\,x^6 - 36\,x^5$ by $9\,x^5$.
5. $15\,x^5 - 25\,x^4$ by $-5\,x^3$.
6. $-24\,x^6 - 32\,x^4$ by $-8\,x^3$.
7. $a^3 - a^2b - a^2b^2$ by a^2.
8. $a^2 - ab - ac$ by $-a$.
9. $xy - x^2 - ax$ by $-x$.
10. $3\,x^3 - 9\,x^2y^2$ by $-3\,x$.
11. $4\,a^4b^4 - 8\,a^3b^3 + 6\,ab^3$ by $-2\,ab$.
12. $-3\,x^2 + \frac{9}{2}\,xy - 6\,xz$ by $-\frac{3}{2}\,x$.
13. $-\frac{5}{2}\,x^2 + \frac{5}{3}\,xy + \frac{10}{3}\,x$ by $-\frac{5}{6}\,x$.
14. $\frac{1}{2}\,x^5y^2 - 3\,x^3y^4 - 5\,x^4y^3$ by $-\frac{3}{2}\,x^3y^2$.

15. $\frac{1}{4}a^2x - \frac{1}{16}abx - \frac{3}{8}acx$ by $\frac{3}{8}ax$.

16. $-2a^5x^3 + \frac{7}{2}a^4x^4$ by $\frac{7}{3}a^3x$.

17. $25(x+y)^3 - 3a(x+y)^2 + 10b(x+y)$ by $5(x+y)$.

18. $-8(a-b)^3 - 12x(a-b)^4 - 16y(a-b)^6$ by $4(a-b)^2$.

19. $6a^{3m} - 4a^{2m}$ by $2a^n$.

20. $10y^{n+2}z^3 - 15y^{n+3}z^2$ by $-5y^nz$.

Divide $12x^{2n+1}y^4 - 16x^{2n+2}y^3 - 20x^{2n+3}y^5$ by:

21. $4x^n$.

22. $-x^{n+1}y^2$.

23. $-8x^{2n}y$.

24. $\frac{1}{4}x^{2n-3}y^2$.

25. Divide $4x^{2n+1}y^a - 16x^{2n+3}y^{a+1}$ by $4x^{2n}y^a$.

26. Divide $-15x^{b+1}y^{a+3} + 21x^{b+2}y^{a+4}$ by $3x^{b+1}y^{a+2}$.

96. To divide one polynomial by another.

Let it be required to divide

$$2x^2y^2 - x^3y + x^4 - xy^3 + y^4 \text{ by } y^2 - xy + x^2.$$

First arrange dividend and divisor in descending powers of x, for convenience placing the divisor to the right of the dividend as below:

$$\begin{array}{r|l}
x^4 - x^3y + 2x^2y^2 - xy^3 + y^4 & x^2 - xy + y^2 \text{ Divisor} \\ \cline{2-2}
\underline{x^4 - x^3y + x^2y^2 } & x^2 + y^2 \quad \text{Quotient} \\
x^2y^2 - xy^3 + y^4 & \\
\underline{x^2y^2 - xy^3 + y^4} &
\end{array}$$

From the law of exponents we know that x^4, the term of the highest degree in x in the dividend, is the product of the terms of highest degree in the divisor and the quotient; hence, the first term of the quotient is $x^4 \div x^2$, or x^2. Multiply the divisor by x^2 and subtract the result from the dividend.

The remainder, $x^2y^2 - xy^3 + y^4$, is the product of the divisor by the other terms of the quotient; hence, x^2y^2, the first term of the remainder, is the product of the first term of the divisor and the second term of the quotient. Therefore the second term of the quotient is $x^2y^2 \div x^2$, or y^2. Multiplying the divisor by y^2 and subtracting the result from $x^2y^2 - xy^3 + y^4$, we have no remainder.

Hence the required quotient is $x^2 + y^2$.

Observe that by the above process the dividend was separated into the two parts $x^4 - x^3y + x^2y^2$ and $x^2y^2 - xy^3 + y^4$; hence, by the distributive law for division, we have

$$\frac{x^4 - x^3y + 2x^2y^2 - xy^3 + y^4}{x^2 - xy + y^2} \equiv \frac{x^4 - x^3y + x^2y^2}{x^2 - xy + y^2} + \frac{x^2y^2 - xy^3 + y^4}{x^2 - xy + y^2}$$
$$\equiv x^2 + y^2.$$

If the dividend and divisor were arranged in ascending powers of x, the quotient would be obtained in the form $y^2 + x^2$.

Hence, to divide one polynomial by another, we have the following rule:

Arrange the dividend and divisor in descending powers of some common letter.

Divide the first term of the dividend by the first term of the divisor, and write the result as the first term of the quotient.

Multiply the divisor by this first term of the quotient, and subtract the resulting product from the dividend.

Divide the first term of the remainder by the first term of the divisor, and write the result as the second term of the quotient.

Multiply the divisor by this second term of the quotient, and subtract the resulting product from the remainder previously obtained.

Treat the second remainder, if any, as a new dividend and go on repeating the process until the remainder is zero, or is of a lower degree in the letter of arrangement than the divisor.

Ex. 1. Divide $2a - 4a^2 + 3a^3 - 1$ by $1 - a$.

Arranging dividend and divisor in descending powers of a, we have

$$\begin{array}{rrrr|l}
3a^3 & -4a^2 & +2a & -1 & -a+1 \\
\cline{5-5}
3a^3 & -3a^2 & & & -3a^2+a- \\
\cline{1-2}
 & -a^2 & +2a & & \\
 & -a^2 & +a & & \\
\cline{2-4}
 & & a & -1 & \\
 & & a & -1 & \\
\cline{3-4}
\end{array}$$

Ex. 2. Divide $x^2y^2 + x^4 + y^4$ by $y^2 - xy + x^2$.

Arranging dividend and divisor in descending powers of x, we have

$$
\begin{array}{llll|l}
x^4 & & +\,x^2y^2 & +\,y^4 & x^2 - xy + y^2 \\ \cline{5-5}
x^4 & -\,x^3y & +\,x^2y^2 & & x^2 + xy + y^2 \\ \cline{1-3}
 & x^3y & & & \\
 & x^3y & -\,x^2y^2 & +\,xy^3 & \\ \cline{2-4}
 & & x^2y^2 & -\,xy^3 + y^4 & \\
 & & x^2y^2 & -\,xy^3 + y^4 & \\ \cline{3-4}
\end{array}
$$

Ex. 3. Divide $16\,a^4 - 1$ by $2\,a - 1$.

$$
\begin{array}{lllll|l}
16\,a^4 & & & & -1 & 2\,a - 1 \\ \cline{6-6}
16\,a^4 & -\,8\,a^3 & & & & 8\,a^3 + 4\,a^2 + 2\,a + 1 \\ \cline{1-2}
 & 8\,a^3 & & & & \\
 & 8\,a^3 & -\,4\,a^2 & & & \\ \cline{2-3}
 & & 4\,a^2 & & & \\
 & & 4\,a^2 & -\,2\,a & & \\ \cline{3-5}
 & & & 2\,a & -\,1 & \\
 & & & 2\,a & -\,1 & \\ \cline{4-5}
\end{array}
$$

Exercise 34.

Divide:

1. $x^2 + 3\,x + 2$ by $x + 1$.
2. $a^2 - 11\,a + 30$ by $a - 5$.
3. $x^2 - 7\,x + 12$ by $x - 3$.
4. $3\,x^2 + 10\,x + 3$ by $x + 3$.
5. $5\,x^2 + 11\,x + 2$ by $x + 2$.
6. $5\,x^2 + 16\,x + 3$ by $x + 3$.
7. $2\,x^2 + 11\,x + 5$ by $2\,x + 1$.
8. $2\,x^2 + 17\,x + 21$ by $2\,x + 3$.
9. $4\,x^2 + 23\,x + 15$ by $4\,x + 3$.
10. $6\,x^2 - 7\,x - 3$ by $2\,x - 3$.
11. $12\,a^2 - 7\,ax - 12\,x^2$ by $3\,a - 4\,x$.
12. $15\,a^2 + 17\,ax - 4\,x^2$ by $3\,a + 4\,x$.
13. $12\,a^2 - 11\,ac - 36\,c^2$ by $4\,a - 9\,c$.
14. $60\,x^2 - 4\,xy - 45\,y^2$ by $10\,x - 9\,y$.

15. $-4xy - 15y^2 + 96x^2$ by $12x - 5y$.

16. $100x^3 - 3x - 13x^2$ by $3 + 25x$.

17. $16 - 96x + 216x^2 - 216x^3 + 81x^4$ by $2 - 3x$.

18. $x^3 - x^2 - 9x - 12$ by $x^2 + 3x + 3$.

19. $2y^3 - 3y^2 - 6y - 1$ by $2y^2 - 5y - 1$.

20. $6m^3 - m^2 - 14m + 3$ by $3m^2 + 4m - 1$.

21. $6a^5 - 13a^4 + 4a^3 + 3a^2$ by $3a^3 - 2a^2 - a$.

22. $x^4 + x^3 + 7x^2 - 6x + 8$ by $x^2 + 2x + 8$.

23. $a^4 - a^3 - 6a^2 + 15a - 9$ by $a^2 + 2a - 3$.

24. $a^4 + 6a^3 + 13a^2 + 12a + 4$ by $a^2 + 3a + 2$.

25. $2x^4 - x^3 + 4x^2 + 4x - 3$ by $x^2 - x + 3$.

26. $x^5 - 5x^4 + 9x^3 - 6x^2 - x + 2$ by $x^2 - 3x + 2$.

27. $x^5 - 4x^4 + 3x^3 + 3x^2 - 3x + 2$ by $x^2 - x - 2$.

28. $30x^4 + 11x^3 - 82x^2 - 12x + 48$ by $2x - 4 + 3x^2$.

29. $69y - 18 - 71y^3 + 28y^4 - 35y^2$ by $4y^2 - 13y + 6$.

30. $6k^5 - 15k^4 + 4k^3 + 7k^2 - 7k + 2$ by $3k^3 - k + 1$.

31. $2x^3 - 8x + x^4 + 12 - 7x^2$ by $x^2 + 2 - 3x$.

32. $x^5 - 2x^4 - 7x^3 + 19x^2 - 10x$ by $x^3 - 7x + 5$.

33. $14x^4 + 45x^3y + 78x^2y^2 + 45xy^3 + 14y^4$ by $2x^2 + 5xy + 7y^2$.

34. $x^5 + x^4y - x^3y^2 + x^3 - 2xy^2 + y^3$ by $x^2 + xy - y^2$.

35. $x^7 - 2y^{14} - 7x^5y^4 - 7xy^{12} + 14x^3y^8$ by $x - 2y^2$.

36. $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.

$$\begin{array}{l|l}
a^3 - 3\,abc + b^3 + c^3 & a + b + c \\
\underline{a^3 + \quad a^2b + a^2c} & a^2 - ab - ac + b^2 - bc + c^2 \\
\quad - a^2b - a^2c - 3\,abc + b^3 + c^3 & \\
\quad \underline{- a^2b - ab^2 - \quad abc} & \\
\qquad - a^2c + ab^2 - 2\,abc + b^3 + c^3 & \\
\qquad \underline{- a^2c \qquad - \quad abc - ac^2} & \\
\qquad\quad ab^2 - abc + ac^2 + b^3 + c^3 & \\
\qquad\quad \underline{ab^2 \qquad\qquad + b^3 + b^2c} & \\
\qquad\qquad - abc + ac^2 - b^2c + c^3 & \\
\qquad\qquad \underline{- abc \qquad - b^2c - bc^2} & \\
\qquad\qquad\qquad ac^2 + bc^2 + c^3 & \\
\qquad\qquad\qquad \underline{ac^2 + bc^2 + c^3} &
\end{array}$$

Here we arranged the dividend and divisor in descending powers of a, and gave b precedence to c throughout.

37. $x^3 + y^3 - z^3 + 3\,xyz$ by $x + y - z$.

38. $8\,x^3 - y^3 + z^3 + 6\,xyz$ by $y - z - 2\,x$.

39. $27\,a^3 - 8\,b^3 + c^3 + 18\,abc$ by $3\,a - 2\,b + c$

40. $a^3 + 3\,a^2b + 3\,ab^2 + b^3 + c^3$ by $a + b + c$.

41. $a^4 + b^4 + c^4 - 2\,b^2c^2 - 2\,a^2c^2 - 2\,a^2b^2$ by $a + b + c$.

42. $\frac{1}{4}\,x^3 + \frac{1}{72}\,xy^2 + \frac{1}{12}\,y^3$ by $\frac{1}{2}\,x + \frac{1}{3}\,y$.

43. $\frac{1}{8}\,a^3 - \frac{9}{4}\,a^2x + \frac{27}{2}\,ax^2 - 27\,x^3$ by $\frac{1}{2}\,a - 3\,x$.

44. $\frac{1}{27}\,a^3 - \frac{1}{12}\,a^2 + \frac{1}{16}\,a - \frac{1}{64}$ by $\frac{1}{3}\,a - \frac{1}{4}$.

45. $\frac{3}{4}\,a^2c^3 + \frac{6}{125}\,a^5$ by $\frac{1}{5}\,a^2 + \frac{1}{2}\,ac$.

46. $\frac{9}{16}\,a^4 - \frac{3}{4}\,a^3 - \frac{7}{4}\,a^2 + \frac{4}{3}\,a + \frac{16}{9}$ by $\frac{3}{2}\,a^2 - \frac{8}{3} - a$.

47. $36\,x^2 + \frac{1}{9}\,y^2 + \frac{1}{4} - 4\,xy - 6\,x + \frac{1}{3}\,y$ by $6\,x - \frac{1}{3}\,y - \frac{1}{2}$.

48. $\frac{8}{27}\,a^5 - \frac{243}{512}\,ax^4$ by $\frac{2}{3}\,a - \frac{3}{4}\,x$.

49. $43\,x^{2m-1} + 6\,x^{2m+1} - 29\,x^{2m} - 20\,x^{2m-2}$ by $2\,x^m - 5\,x^{m-1}$.

$$\begin{array}{l|l}
6\,x^{2m+1} - 29\,x^{2m} + 43\,x^{2m-1} - 20\,x^{2m-2} & 2\,x^m - 5\,x^{m-1} \\
\underline{6\,x^{2m+1} - 15\,x^{2m} \qquad\qquad} & 3\,x^{m+1} - 7\,x^m + 4\,x^{m-1} \\
\qquad - 14\,x^{2m} + 43\,x^{2m-1} & \\
\qquad \underline{- 14\,x^{2m} + 35\,x^{2m-1}} & \\
\qquad\qquad\qquad 8\,x^{2m-1} - 20\,x^{2m-2} & \\
\qquad\qquad\qquad \underline{8\,x^{2m-1} - 20\,x^{2m-2}} &
\end{array}$$

50. $x^{4n}-x^{3n}y^m+x^ny^{3m}-y^{4m}$ by x^n-y^m.

51. $6a^{3n}-25a^{2n}+27a^n-5$ by $2a^n-5$.

52. $6a^{5n}-11a^{4n}+13a^{2n}+23a^{3n}+2-3a^n$ by $3a^n+2$.

53. $12x^{n+1}+8x^n-45x^{n-1}+25x^{n-2}$ by $6a-5$.

54. $\frac{15}{2}a^2-\frac{113}{6}ab+9ac+2b^2-bc$ by $\frac{5}{4}a-3b+\frac{3}{2}c$.

55. $(b+c)x^2-bcx+x^3-bc(b+c)$ by x^2-bc.

56. $x^3+(a+b+c)x^2+(ab+ac+bc)x+abc$ by $x+b$.

57. $x^3+(a+b-c)x^2+(ab-ac-bc)x-abc$ by $x-c$.

97. When, as in each example given above, the division is exact, the quotient is the same whether the dividend and divisor are arranged in *descending* or in *ascending* powers of any common letter. But when the division is not exact, the *partial* quotient obtained with one arrangement is not the same as that obtained with the other.

E.g.,
$$\frac{x^2+1}{x+1}\equiv x-1+\frac{2}{x+1}; \tag{1}$$

while
$$\frac{1+x^2}{1+x}\equiv 1-x+2x^2-2x^3+\frac{2x^4}{1+x}. \tag{2}$$

Here the *partial* quotients $x-1$ and $1-x+2x^2-2x^3$ are evidently unequal. The *entire* quotients, or the second members of (1) and (2), are, of course, identical.

In (1) the remainder is of a lower degree than the divisor.

In (2) the division can be carried to any number of terms.

When arranged in ascending powers of some common letter, an expression of a lower degree can be divided by one of a higher degree in the letter of arrangement.

$$E.g.,\ \frac{1}{1-x}\equiv 1+x+x^2+x^3+\cdots+x^{n-1}+\frac{x^n}{1-x}.$$

Exercise 35.

Divide:

1. x^2+y^2 by $x+y$.
2. x^3+y^3 by $x-y$.
3. x^3-y^3 by $x+y$.
4. x^4+y^4 by $x+y$.
5. x^5-a^5 by $x+a$.
6. 1 by $1+x$ to 4 terms.
7. $1+x$ by $1+x^2$ to 5 terms.
8. $1+2x$ by $1-3x$ to 4 terms.

9. Divide 2 by $1+x$ and thus reduce the second member of identity (1) in § 97 to the form of the second member of (2).

98. *Zero divided by any number,* **except zero,** *is equal to zero.*

That is, when $a \neq 0,\ 0 \div a \equiv 0.$ §§ 85, 74

Conversely, *if a quotient is zero, the dividend is zero.*

Zero as divisor will be considered in Chapter XXVII. Prior to that chapter it will be assumed that any expression used as a divisor does not denote zero.

99. Division by detached coefficients. In § 82 we considered two cases in which the work of multiplication could be shortened by using the **method of detached coefficients.** In the same two cases the labor of division can be lessened by using detached coefficients and an arrangement of terms known as **Horner's method of synthetic division.** This method is illustrated by the following examples:

Ex. 1. Divide $2x^5-7x^4+2x^3-x^2-6x+20$ by $2x^3-3x^2+4x-5$.

Modified Divisor							
2	2	−7	+2	−1	−6	+20	Dividend
3		3	−4	5			(1)
−4			−6	+8	−10		(2)
5				−12	+16	−20	(3)
Quotient	1	−2	−4	0	0	0	Remainder

Inserting in the quotient the literal factors, whose law of formation is seen by inspection, we have for the complete quotient x^2-2x-4.

Explanation. The *modified divisor*, or the column of figures to the left of the vertical line, consists of the coefficients of the divisor, *the*

quality of each coefficient after the first being changed; this change of quality enables us *to replace the operation of subtraction by that of addition* at each successive stage of the work.

Observe that the number of coefficients in the quotient will be one more than the number of coefficients in the dividend minus the number of coefficients in the divisor, in this case $1 + 6 - 4$, or 3 (§ 82). Thus, the numbers to the left of the vertical bar are the coefficients of the *quotient*, and those to the right of this bar are the coefficients of the remainder.

Dividing the first coefficient of the dividend by the first coefficient of the divisor, we obtain the first coefficient, 1, of the quotient. Multiplying the modified coefficients of the divisor $(3, -4, 5)$ by this first coefficient of the quotient, we obtain line (1).

Adding the coefficients in the *second* column to the right of the divisor, and dividing the sum by the first coefficient, 2, of the divisor, we obtain -2, which is the second coefficient of the quotient. Multiplying the modified coefficients of the divisor by this *second* coefficient of the quotient, we obtain line (2).

Adding the coefficients in the *third* column and dividing the sum by the first coefficient of the divisor, we obtain the *third*, or last, coefficient of the quotient. Multiplying the modified coefficients of the divisor by this third coefficient of the quotient, we obtain line (3).

Lines (1), (2), and (3) are evidently the coefficients of the three partial products obtained by multiplying the divisor by each term of the quotient, the first term of each product being omitted and the quality of the others being changed.

Hence by adding each of the vertical columns after the third, we obtain the coefficients of the remainder.

Here the coefficients are all zero, and the division is exact.

Ex. 2. Divide $2x^6 - 7x^5y + 12x^4y^2 - 8x^3y^3 + x^2y^4$ by $2x^3 - 3x^2y - y^3$.

$$
\begin{array}{r|rrrr|rrrl}
2 & 2 & -7 & +12 & -8 & +1 & +0 & +0 & \quad (1) \\
3 & & +3 & 0 & +1 & & & & \quad (2) \\
0 & & & -6 & 0 & -2 & & & \quad (3) \\
1 & & & & +9 & 0 & +3 & & \quad (4) \\
 & & & & & +3 & 0 & +1 & \quad (5) \\
\hline
 & 1 & -2 & +3 & +1 & +2 & +3 & +1 &
\end{array}
$$

Inserting the literal factors, we have for the *quotient* $x^3 - 2x^2y + 3xy^2 + y^3$, and for the remainder $2x^2y^4 + 3xy^5 + y^6$.

Explanation. The terms in xy^5 and y^6 are missing in the dividend, and the term in xy^2 in the divisor; hence their zero coefficients are

written with the other coefficients. The sums of the vertical columns after the fourth give the coefficients of the remainder.

To find the remainder after one term of the quotient, add lines (1) and (2) after the first vertical column; to find the remainder after two terms of the quotient, add lines (1), (2), and (3) after the first two vertical columns; to find the remainder after three terms of the quotient, add lines (1), (2), (3), and (4) after the first three vertical columns.

Exercise 36.

Divide:

1. $x^4 - 4x^3 + 2x^2 + 4x + 1$ by $x^2 - 2x - 1$.

2. $2x^7 + x^6 + 2x^5 + 1$ by $1 + x + x^2$.

3. $x^{10} + x^5 + 1$ by $x^2 + x + 1$.

4. $6x^5 - 7x^4y + x^3y^2 + 20x^2y^3 - 22xy^4 + 8y^5$
   by $2x^2 - 3xy + 4y^2$.

5. $8y^5 - 22xy^4 + 20x^2y^3 + x^3y^2 - 7x^4y + 6x^5$
   by $4y^2 - 3xy + 2x^2$.

6. $a^5 - 3a^2b^3 + 8ab^4 - 5b^5$ by $a^2 - 4ab + b^2$ to four terms in the quotient.

CHAPTER VII

INTEGRAL LINEAR EQUATIONS IN ONE UNKNOWN

100. An **integral equation** is an equation all of whose terms are integral in the unknown. (Review §§ 10, 16, 17.)

E.g., $2x^2 + 3 = 2x$ and $\frac{x^2}{2} + \frac{ax}{b} = x + 2$ are integral equations.

101. The **degree** of an *integral* equation in one unknown is the degree of its term of highest degree in the unknown.

A **linear equation** is an equation of the *first* degree.

A **quadratic equation** is an equation of the *second* degree.

A **higher equation** is an equation of a *higher* degree than the second.

E.g., $3x + 1 = 4$ and $ax + b = 0$ are *linear* equations in x.

$5x^2 - 7x = 1$ and $ax^2 + bx + c = 0$ are *quadratic* equations in x.

$6x^3 - 4x^2 + 3x + 4 = 0$ is a *higher* equation in x.

102. A **root**, or *solution*, of an equation in one unknown is any *value of the unknown;* that is, it is any number which when substituted for the unknown renders the equation an identity.

E.g., 12 is a *root* of the equation

$$2x - 5 = x + 7.$$

For, putting 12 for x in the equation, we obtain the identity

$$24 - 5 = 12 + 7.$$

Any root of an equation, since it satisfies the condition expressed by the equation, is said *to satisfy the equation.*

103. To **solve** an equation in one unknown is to find *all* its roots. In solving equations we use the principles of

EQUIVALENT EQUATIONS.

104. Two equations in one unknown are said to be **equivalent**, when every root of the first is a root of the second, and every root of the second is a root of the first.

E.g., the equations

$$4x - 8 = 2 - x \tag{1}$$

and

$$5x = 10 \tag{2}$$

have the same root, *i.e.*, are *equivalent;* for 2 is a *root* of each equation, and, as will be seen later, 2 is the *only* root of either.

In solving equations we need to know what operations on the members of an equation will make the *derived* equation have the same root, or roots, as the given one.

Of such operations the most elementary and important are found in §§ 105, 106, 108, 109.

105. Identical expressions.

If in the equation

$$4(x-1)-(3x-2)=3, \tag{1}$$

we substitute for the first member the identical expression $x-2$, we obtain the *equivalent* equation

$$x-2=3. \tag{2}$$

For, as is easily shown, 5 is a root of either equation; and, as will be seen later, 5 is the *only* root of either.

This example illustrates the following principle:

If, for any expression in an equation, an identical expression is substituted, the derived equation will be equivalent to the given one.

That is, if $A = B$ denotes any equation in one unknown, as x, and $A \equiv A'$; then the equations

$$A = B \tag{1}$$

and

$$A' = B \tag{2}$$

have the same root, or roots.

Proof. To prove that equations (1) and (2) have the same root, or roots, we must prove that every root of (1) is a root of (2); and conversely that every root of (2) is a root of (1).

Since A and A' are identical expressions, any value of x which when substituted for x will make either one identical with B, will make the other identical with B (§ 32). Hence, any root of (1) is a root of (2), (§ 102); and conversely any root of (2) is a root of (1); that is, equations (1) and (2) have the same roots, *i.e.*, are equivalent.

E.g., since, $3(x-1)-\{3x-(2+x)\} \equiv x-1$;

the equations

$$3(x-1)-\{3x-(2+x)\} = 5 \tag{1}$$

and

$$x - 1 = 5 \tag{2}$$

have the same root; that is, we neither lose nor introduce a root by substituting for $3(x-1)-\{3x-(2+x)\}$ in equation (1) its *identical* expression $x-1$.

106. Addition or Subtraction.

If to both members of the equation

$$2x - 8 = 7 - 8 \tag{1}$$

we add $8 + x$, we obtain the equivalent equation

$$3x = 15.$$

For, as is easily shown, 5 is a root of each equation; and, as will be seen later, 5 is the *only* root of either equation.

This example illustrates the following principle:

If identical expressions are added to, or subtracted from, both members of an equation, the derived equation will be equivalent to the given one.

That is, if $M \equiv M'$, the equations

$$A = B \tag{1}$$

and

$$A \pm M = B \pm M' \tag{2}$$

have the same root or roots.

Proof. Any root of (1) makes $A \equiv B$. § 102

But, when $A \equiv B$, $A \pm M \equiv B \pm M'$. § 32, (iii)

Hence, any root of (1) is a root of (2).

Conversely, any root of (2) makes $A \pm M \equiv B \pm M'$.

But, when $A \pm M \equiv B \pm M'$, $A \equiv B$. § 32, (iii)

Hence, any root of (2) is a root of (1).

Therefore, equations (1) and (2) are equivalent. § 104

If, to each member of the equation

$$ax - b = cx - d, \tag{1}$$

we add $-cx$ and $+b$, we obtain the equivalent equation

$$ax - cx = b - d. \tag{2}$$

Adding $-cx$ to both members of equation (1) removes the term $+cx$ from the second member, and transfers it, with its sign changed from $+$ to $-$, to the first member. Likewise, adding $+b$ to both members of (1) removes the term $-b$ from the first member, and transfers it, with its sign changed from $-$ to $+$, to the second member. This example illustrates the following important application of the principle proved above.

If any term is transposed from one member of an equation to the other, its sign being changed from $+$ to $-$, or from $-$ to $+$, the derived equation has the same root or roots as the given one.

107. An expression is said to be *unknown*, or *known*, according as it *does*, or *does not*, contain an unknown number.

E.g., if x is an *unknown* number, $x - 2$ is an *unknown* expression; if a is a *known* number, $9 + 5a$ is a *known* expression.

108. Multiplication.

If both members of the equation

$$\frac{x}{2}+\frac{3}{4}=\frac{x}{3}+\frac{13}{12} \tag{1}$$

are multiplied by 12, we obtain the equivalent equation

$$6x+9=4x+13. \tag{2}$$

For, as is easily shown, 2 is a root of each equation, and, as will be seen later, 2 is the only root of either.

This example illustrates the following principle:

If both members of an equation are multiplied by the same known expression, not denoting zero, the derived equation will be equivalent to the given one.

That is, if C represents any *known* expression, *not denoting zero,* the equations

$$A=B \tag{1}$$

and

$$CA=CB \tag{2}$$

have the same roots.

Proof. Any root of (1) makes $A\equiv B$. § 102

But, when $A\equiv B$, $CA\equiv CB$. § 32, (iv)

Hence, any root of (1) is a root of (2).

Conversely, any root of (2) makes $CA\equiv CB$. § 102

But, when $CA\equiv CB$, $A\equiv B$, since $C\neq 0$. § 32, (v)

Hence, any root of (2) is a root of (1).

Therefore, equations (1) and (2) are equivalent.

Ex. 1. Solve the equation $(5x-12)\div 6=(x-3)\div 3.$ (1)

Multiply by 6, $5x-12=2x-6.$ (2)

Transpose terms, $5x-2x=12-6.$ (3)

Unite terms, $3x=6.$ (4)

Multiply by 1/3, $x=2.$ (5)

Proof of equivalency.

Equation (2) has the same roots as (1) by § 108, 'identical expressions.'

Equation (3) has the same roots as (2) by § 106, 'addition.'

Equation (4) has the same roots as (3) by § 105, 'identical expressions.'

Equation (5) has the same roots as (4) by § 108, 'multiplication.'

Hence the one and only root of each of these equations is 2.

Ex. 2. Solve the equation $\frac{x+1}{4} - \frac{x-1}{3} = 1.$ (1)

Multiply by 12, $3(x+1) - 4(x-1) = 12.$ (2)

Remove ( ), $3x + 3 - 4x + 4 = 12.$ (3)

Transpose terms, $3x - 4x = 12 - 3 - 4.$ (4)

Unite terms, $-x = 5.$ (5)

Multiply by -1, $x = -5.$ (6)

Proof of equivalency. Equation (2) has the same roots as (1) by the principle of 'multiplication' (§ 108); (3) as (2) by 'identical expressions' (§ 105); (4) as (3) by 'addition' (§ 106); (5) as (4) by 'identical expressions' (§ 105); and (6) as (5) by 'multiplication' (§ 108). Hence the one and only root of each of these equations is -5.

The two following applications of the foregoing principle are very important:

(i) When, to clear an equation of *fractional coefficients*, we multiply both members by the L.C.M. of their *known* denominators, the derived equation has the same roots as the given one.

(ii) When the sign before each term of an equation is changed from + to −, or from − to + (that is, when each member is multiplied by -1) the derived equation has the same roots as the given one.

109. Roots introduced or lost.

If we multiply both members of the *equation*

$$3x - 7 = 2x + 2 \qquad (1)$$

by the known expression 0, we obtain the *identity*

$$(3x - 7) \times 0 \equiv (2x + 2) \times 0. \qquad (2)$$

Observe that (1) restricts x to the one value 9, while (2) does not restrict the value of x at all.

Again, if we multiply both members of the equation

$$6x - 1 = 4x + 3 \tag{3}$$

by the unknown expression $x - 5$, we obtain the equation

$$(6x - 1)(x - 5) = (4x + 3)(x - 5). \tag{4}$$

Equation (4) has the two roots 2 and 5, while (3) has only the one root 2. Hence, the root 5 was *introduced* into (4) by multiplying (3) by the *unknown* expression $x - 5$.

The two examples above illustrate why the multiplier in § 108 was limited to a *known* expression, *not denoting zero.*

If we divided *identity* (2) by 0, we would obtain *equation* (1).

If we divided equation (4) by $x - 5$, we would obtain equation (3), and one root would be *lost* by the operation.

This illustrates why the divisor in the following article is limited to a *known* expression *not denoting zero.*

110. Division. *If both members of an equation are divided by the same known expression, not denoting zero, the derived equation will be equivalent to the given one.*

That is, if C represents any *known* expression, not denoting zero, the equations

$$A = B \tag{1}$$

and

$$A \div C = B \div C \tag{2}$$

have the same roots.

Proof. Any root of (1) makes $A \equiv B$. § 102

But, when $A \equiv B$, $A \div C \equiv B \div C$. § 32, (v)

Hence, any root of (1) is a root of (2).

Conversely, any root of (2) makes $A \div C \equiv B \div C$. § 102

But, when $A \div C \equiv B \div C$, $A \equiv B$. § 32, (iv)

Hence, any root of (2) is a root of (1).

Therefore, equations (1) and (2) are equivalent.

Ex. 1. Solve the equation $(x-1)(x-2)+5=(x+1)^2$. (1)

Remove ( ), $x^2-3x+2+5=x^2+2x+1$. (2)

Transpose terms, $-3x-2x=1-2-5$. (3)

Unite terms, $-5x=-6$. (4)

Divide by -5, $x=\frac{6}{5}$. (5)

Proof of equivalency. No root is lost or introduced by any one of the operations performed on the members of the equations from (1) to (5); hence, the one and only root of (1) is $\frac{6}{5}$.

Ex. 2. Solve $3(x-1)-\{3x-(2-x)\}=5$. (1)

Remove ( ), $3x-3-3x+2-x=5$. (2)

Transpose, $3x-3x-x=5+3-2$ (3)

Unite terms, $-x=6$. (4)

Divide by -1, $x=-6$. (5)

Proof of equivalency. No root is lost or introduced by any one of the operations performed on the members of the equations from (1) to (5); hence, the one and only root of (1) is -6.

Observe the difference in the meanings of *equal*, *identical*, and *equivalent*. *Equal* applies to *numbers*, *identical* to *expressions*, and *equivalent* to *equations*. Two numbers are equal or unequal, two expressions are identical or not identical, and two equations are equivalent or not equivalent, *i.e.*, have or have not the same roots. We should not apply the word *equivalent* to numbers or expressions.

111. From the examples given above, it will be seen that the different steps in the process of solving a linear equation are the following:

(i) *Clear the equation of fractions, if there are any.*

(ii) *Remove parentheses, if there are any.*

(iii) *Transpose the unknown terms to one member of the equation, and the known terms to the other member.*

(iv) *Unite like terms, and divide both members by the coefficient of the unknown.*

NOTE. In order to form the habit of clear and accurate thinking, the pupil should at first state the operation by which each equation is derived from the preceding one, and note whether by this operation any root is lost or introduced.

But as he advances he should perform the simpler steps mentally, and apply two or more principles at the same time.

112. A **numeral, or numerical, equation** is an equation in which all the known numbers are denoted by numerals.

Exercise 37.

Solve each of the following numeral equations:

1. $3x - 5 = 2x + 1$.
2. $3x + 4 = x + 8$.
3. $4x - 4 = x - 7$.
4. $7x - 5 = x - 23$.
5. $8x + 42 = 5x$.
6. $5x - 12 = 6x - 8$.
7. $7x + 19 = 5x + 7$.
8. $x - (4 - 2x) = 7(x - 1)$.
9. $5(4 - 3x) = 7(3 - 4x)$.
10. $4(1 - x) + 3(2 + x) = 13$.
11. $3(x - 2) = 2(x - 3)$.
12. $2x - (5x + 5) = 7$.
13. $3(x + 1) = -5(x - 1)$.
14. $7(x - 18) = 3(x - 14)$.
15. $2(x - 2) + 3(x - 3) + 4(x - 4) - 20 = 0$.
16. $2(x - 1) - 3(x - 2) + 4(x - 3) + 2 = 0$.
17. $5x + 6(x + 1) - 7(x + 2) - 8(x + 3) = 0$.
18. $2x - [3 - \{4x + (x - 1)\} - 5] = 8$.
19. $(x - 1)(x - 2) = (x + 3)(x - 4)$.
20. $3x^2 = (x + 1)^2 + (x + 2)^2 + (x + 3)^2$.
21. $(x - 2)(x - 5) + (x - 3)(x - 4) = 2(x - 4)(x - 5)$.
22. $5(x + 1)^2 + 7(x + 3)^2 = 12(x + 2)^2$.
23. $(x - 1)(x - 4) = 2x + (x - 2)(x - 3)$.

24. $x/5 - x/4 = 1.$

25. $(x-1)/2+(x-2)/3=3.$

26. $2x/3 + 4 = 5 + x/3.$

27. $2x/3 + 5 = 5x/6 + 4.$

28. $x/2+2x/3=5x/6+7.$

29. $3x/4 - 5 = 7x/8 - 6.$

30. $\frac{1}{2}(2 - x) - \frac{1}{5}(5x + 21) = x + 3.$

31. $\frac{1}{2}(x + 1) + \frac{1}{3}(x + 2) + \frac{1}{4}(x + 4) + 8 = 0.$

32. $\frac{1}{2}(x - 5) - \frac{1}{3}(x - 4) = \frac{1}{2}(x - 3) - (x - 2).$

33. $\frac{1}{2}(x + \frac{1}{2}) - \frac{1}{5}(2x - \frac{1}{2}) + 1\frac{1}{4} = 0.$

34. $(3x + 5)/8 - (21 + x)/2 = 5x - 15.$

35. $(x - 2)/3 - (12 - x)/2 = (5x - 36)/4 - 1.$

36. $(x + 8)/4 - (5x + 2)/3 = (14 - x)/2 - 2.$

37. $(x - 15)/4 - (7 - 2x)/21 = 3x/14 + 1/2.$

38. $5[4 - (3x - 1)] = 6(x - 11) + 49.$

39. $(x - 2)/4 + 1/3 - [x - (2x - 1)/3] = 0.$

40. $3\frac{1}{3}[28 - (x/8 + 24)] = 3\frac{1}{2}(2\frac{1}{3} + x/4).$

41. $5(3x - 5) - 17 = 8(3x - 5) - 2(3x - 5).$

Combine the terms involving $3x - 5$.

42. $2(x + 1) - 3(x + 1) + 9(x + 1) + 18 = 7(x + 1).$

43. $x(x + 2) + x(x + 1) = (2x - 1)(x + 3).$

44. $x^2 - x[1 - x - 2(3 - x)] = x + 1.$

45. $3(x - 1)/16 - 5(x - 4)/12 = 2(x - 6)/5 + 5/48.$

46. $0.5x + 3.75 = 5.25x - 1.$

To clear of fractions multiply by 4.

47. $2.25x - 0.125 = 3x + 3.75.$

48. $0.25x + 4 - 0.375x = 0.2x - 9.$

49. $0.375x - 1.875 = 0.12x + 1.185.$

50. $0.15x + 1.2 - 0.875x + 0.375 = 0.0625x.$

113. A **literal equation** is an equation in which one or more of the *known* numbers are denoted by letters.

E.g., $ax + 2x + 4 = 0$ and $ax + b = cx$ are literal equations.

Ex. Solve the equation $(2 - 5x)/a = (cx + 7)/b$. (1)

Multiply by ab, $(2 - 5x)b = (cx + 7)a$. (2)

Remove ( ), $2b - 5bx = acx + 7a$. (3)

Transpose, $5bx - acx = 7a - 2b$. (4)

Unite terms, $-(5\mathrm{b} + ac)x = 7a - 2b$. (5)

Divide by $-(5b + ac)$, $x = \dfrac{2b - 7a}{5b + ac}$. (6)

Proof of equivalency. No root is lost or introduced by any one of the operations performed on the members of the equations from (1) to (6); hence, the one and only root of (1) is given in (6).

114. *Any linear equation in one unknown has one, and only one, root.*

Proof. By transposing and combining terms, any linear equation can be reduced to an equation of the form

$$ax = c, \qquad (1)$$

which, by §§ 105, 106, is equivalent to the given equation.

Divide by a, $x = c/a$. (2)

Equation (2) is equivalent to (1) by § 110; hence, c/a is the one and only root of equation (1), or of the given linear equation.

If $c = 0$ and $a \neq 0$, the root c/a is zero (§ 98).

If $c \neq 0$, then the smaller a is, the larger arithmetically is the root c/a.

Observe that, if b is an arithmetic number, the linear equation $x - b = 0$ has the arithmetic root b, while the equation $x + b = 0$ has no arithmetic root.

Exercise 38.

Solve each of the following literal equations:

1. $ax + b^2 = bx + a^2$.
2. $x/a + x/b = c$.
3. $(a + b)x + (b - a)x = b^2$.
4. $2(x - a) + 3(x - 2a) = 2a$.
5. $(a + b)x + (a - b)x = a^2$.
6. $(a + bx)(b + ax) = ab(x^2 - 1)$.
7. $(a - x)(a + x) = 2a^2 + 2ax - x^2$.
8. $\frac{1}{2}(x + a + b) + \frac{1}{3}(x + a - b) = b$.
9. $\frac{1}{2}(a + x) + \frac{1}{3}(2a + x) + \frac{1}{4}(3a + x) = 3a$.
10. $xa \div b + xb \div a = a^2 + b^2$.
11. $(a + bx)(b + ax) = ab(x^2 - 1)$.
12. $(a^2 + x)(b^2 + x) = (ab + x)^2$.
13. $(x + a + b)^2 + (x + a - b)^2 = 2x^2$.
14. $(x - a)(x - b) + (a + b)^2 = (x + a)(x + b)$.
15. $ax(x + a) + bx(x + b) = (a + b)(x + a)(x + b)$.
16. What kind of a number is the root of a *numeral* equation? Of a *literal* equation?

See § 5.

CHAPTER VIII

PROBLEMS SOLVED BY LINEAR EQUATIONS IN ONE UNKNOWN

115. Having learned some of the properties of *linear equations in one unknown*, we return to the subject of *solving problems by equations*, which was introduced in the first chapter. Reread § 19.

Prob. 1. A has \$80, and B has \$15. How much must A give to B in order that he may have just 4 times as much as B?

Let $\qquad x =$ the *number* of dollars that A must give to B;

then $\qquad 80 - x =$ the *number* of dollars that A will have left,

and $\qquad 15 + x =$ the *number* of dollars that B will have.

But A will then have 4 times as much as B; that is,

$$80 - x = 4(15 + x). \tag{1}$$

From (1) $\qquad x = 4.$

Hence A must give \$4 to B.

Prob. 2. A man has 16 coins, some of which are half-dollars, and the rest dimes, and the coins altogether are worth \$6. How many has he of each kind?

Let $\qquad x =$ the *number* of half-dollars;

then $\qquad 16 - x =$ the *number* of dimes.

The $\qquad x$ half-dollars are worth $\frac{1}{2}x$ dollars,

and the $\qquad 16 - x$ dimes are worth $\frac{1}{10}(16 - x)$ dollars.

Now the coins altogether are worth \$6; hence

$$\tfrac{1}{2}x + \tfrac{1}{10}(16 - x) = 6.$$

From (1) $\qquad x = 11$, the *number* of half-dollars.

$\therefore\ 16 - x = 5$, the *number* of dimes.

NOTE. It should be remembered that any letter as x always denotes a *number*, and not a concrete quantity. Observe, also, that in any problem all concrete quantities of the same kind must be expressed in terms of the same unit; for example, in each of the above examples all sums of money were expressed in terms of the unit, one dollar.

Prob. 3. A father is 7 times as old as his son, and in 5 years he will be 4 times as old as his son. How old is each?

Let x years = the son's age;

then $7x$ years = the father's age.

Hence $(x+5)$ years = the son's age after 5 years,

and $(7x+5)$ years = the father's age after 5 years.

$$\therefore 7x+5=4(x+5). \qquad (1)$$

From (1) $x=5. \quad \therefore 7x=35.$

Hence the son is 5 years old, and the father 35.

Prob. 4. Divide 60 into two parts, so that three times the greater may exceed 100 by as much as 8 times the less falls short of 200.

Let x = the greater part; then $60-x$ = the less.

Three times the greater part is $3x$, and

$3x-100$ = the excess of $3x$ over 100.

Eight times the less part $=8(60-x)$ and

$200-8(60-x)$ = the defect of $8(60-x)$ from 200.

But this excess and defect are equal; that is,

$$3x-100=200-8(60-x). \qquad (1)$$

From (1) $x=36$, the greater number.

$\therefore 60-x=24$, the less number.

Prob. 5. A could do a piece of work in 14 hours which B could do in 6 hours. A began the work, but after a time B took his place, and the whole work was finished in 10 hours from the beginning. How long did A work?

Let x = the *number* of hours that A worked;

then $10-x$ = the *number* of hours that B worked.

Florian Cajori

Since A could do the whole work in 14 hours, in 1 hour he would do 1/14 of it; hence

$\frac{1}{14}x =$ the part of the work done by A in x hours.

Since B could do the whole work in 6 hours, in 1 hour he would do 1/6 of it; hence

$\frac{1}{6}(10-x) =$ the part of the work done by B in $10-x$ hours.

But A and B together did the whole work; hence

$$\frac{1}{14}x + \frac{1}{6}(10-x) = 1. \qquad (1)$$

From (1) $\quad x = 7. \quad \therefore\ 10 - x = 3.$

Hence A worked 7 hours, and B worked 3.

Prob. 6. Find the time between 5 and 6 o'clock at which the hands of a watch are together.

Suppose that the hands are together at x minutes after 5 o'clock.

At 5 o'clock the hour-hand is 25 minute-spaces ahead of the minute-hand; hence, while the minute-hand moves through x minute-spaces, the hour-hand will move through $x-25$ such spaces. But the minute-hand moves 12 times as fast as the hour-hand; that is, in any given time the minute-hand passes over 12 times as many minute-spaces as the hour-hand. Hence

$$x = 12(x-25). \qquad (1)$$

From (1) $\quad x = 27\frac{3}{11}.$

Hence the hands are together at $27\frac{3}{11}$ minutes past 5 o'clock.

Exercise 39.

1. Find two numbers whose sum is 72, and whose difference is 8. *Ans.* 40 and 32.

2. Divide 25 into two parts whose difference is 5.

3. Divide 12 into two parts whose difference is 16. *Ans.* 14 and -2.

4. The difference between two numbers is 8; if 2 be added to the greater, the result will be 3 times the smaller. Find the numbers.

5. A man walks 12 miles, then travels a certain distance by train, and then twice as far by coach as by train. If the whole journey is 78 miles, how far does he travel by train?

6. Find two numbers whose difference is 12, and whose sum is equal to $\frac{1}{2}$ their difference.

7. Find a number such that the sum of its sixth and ninth parts will be equal to 15.

8. Find the number of which the eighth, sixth, and fourth parts together make up -13. *Ans.* -24.

9. Find a number such that $\frac{4}{5}$ of it shall exceed $\frac{6}{7}$ of it by 2. *Ans.* -35.

10. Two numbers differ by 28, and one is $\frac{8}{9}$ of the other. Find them.

11. A, B, and C have a certain sum of money between them. A has $\frac{1}{2}$ of the whole, B has $\frac{1}{3}$ of the whole, and C has \$50. How much have A and B?

12. A and B together have \$75, and A has \$5 more than B. How much has each?

13. A has \$5 more than B, B has \$20 more than C, and they have \$360 between them. How much has each?

14. A has \$15 more than B, B has \$5 less than C, and they have \$65 between them. How much has each?

15. A has \$100, and B has \$20. How much must A give B in order that B may have half as much as A?

16. The sum of two numbers is 38, and one of them exceeds twice the other by 2. What are the numbers?

17. Find a number which when multiplied by 8 exceeds 27 as much as 27 exceeds the original number.

18. Find two numbers of which the sum is 31, and which are such that one of them is less by 2 than half the other.

19. Divide 100 into two parts such that twice one part is equal to 3 times the other.

20. Four times the difference between the fourth and fifth parts of a certain number exceeds by 4 the difference between the third and seventh parts. Find the number.

21. Fifty times the difference between the seventh and eighth parts of a certain number exceeds half the number by 44. Find the number.

22. A father is 4 times as old as his son; in 24 years he will be only twice as old. Find their ages.

23. A is 25 years older than B, and A's age is as much above 20 as B's is below 85. Find their ages.

24. A's age is 6 times B's, and 15 years hence A will be 3 times as old as B. Find their ages.

25. Find a number such that if 5, 75, and 35 are added to it, the product of the first and third sums will be equal to the second sum multiplied by the number.

26. The difference between the squares of two consecutive whole numbers is 121. Find the numbers.

27. Divide \$380 between A, B, and C, so that B will have \$30 more than A, and C will have \$20 more than B.

28. The sum of the ages of A and B is 30 years, and 5 years hence A will be 3 times as old as B. Find their present ages.

29. The length of a room exceeds its breadth by 3 feet; if the length had been increased by 3 feet, and the breadth diminished by 2 feet, the area would not have been altered. Find the length and breadth of the room.

30. The length of a room exceeds its breadth by 8 feet; if each had been increased by 2 feet, the area would have been increased by 60 square feet. Find the dimensions of the room.

31. The width of a room is $\frac{2}{3}$ of its length. If the width had been 3 feet more, and the length 3 feet less, the room would have been square. Find the dimensions of the room.

32. A, B, and C have \$1285 between them; A's share is greater than $\frac{5}{6}$ of B's by \$25, and C's is $\frac{4}{15}$ of B's. Find the share of each.

33. If silk costs 6 times as much as linen, and I spend \$66 in buying 40 yards of silk and 24 yards of linen, find the cost of each per yard.

34. If \$600 be divided among 10 men, 20 women, and 40 children, so that each man receives \$15 more than each child, and each woman receives as much as two children, find what each receives.

35. Divide \$152 among 5 men, 7 women, and 30 children, giving to each man \$4 more than to each woman, and to each woman 3 times as much as to each child.

36. A sum of money is divided between three persons, A, B, and C, so that A and B have \$60 between them, A and C have \$65, and B and C have \$75. How much has each?

37. A dealer bought four horses for \$1150; the second cost him \$60 more than the first, the third \$30 more than the second, and the fourth \$10 more than the third. How much did each cost?

38. Two coaches start at the same time from York and London, a distance of 200 miles, travelling one at $9\frac{1}{2}$ miles an hour, the other at $9\frac{1}{4}$ miles an hour. In how many hours after starting did they meet, and how far from London?

Ans. $10\frac{2}{3}$ hours; $98\frac{2}{3}$ miles from London.

39. A man leaves $\frac{1}{2}$ of his property to his wife, $\frac{1}{3}$ to his son, and the remainder, which is \$2500, to his daughter. How much did he leave to his wife and son each?

Let $x =$ the *number* of dollars which he left in all.

40. A man divided his property between his three children so that the eldest received twice as much as the second, and the second twice as much as the youngest. The eldest received \$3750 more than the youngest. How much did each receive?

41. A third of the length of a post is in the mud, a fourth is in the water, and 5 feet is above the water. Find the length of the post.

42. A flock of sheep and goats together number 84. There are 3 goats to every 4 sheep. How many are there of each?

43. Find the time between 3 and 4 at which the hands of a clock are together.

44. A can do a piece of work in 30 days which B can do in 20 days. A begins the work, but after a time B takes his place, and the whole work is finished in 25 days from the beginning. How long did A work?

45. A can do a piece of work in 20 days which B can do in 30 days. A begins work, but after a time B takes his place and finishes it. B worked 10 days longer than A. How long did A work?

46. One number exceeds another by 3, while its square exceeds the square of the second by 99. Find the numbers.

47. Of two consecutive numbers, $\frac{1}{5}$ of the greater exceeds $\frac{1}{7}$ of the less by 3. Find the numbers.

48. A garrison of 1000 men having provisions for 60 days was reinforced after 10 days, and from that time the provisions lasted only 20 days. Find the number in the reinforcement.

49. In a mixture of spirits and water half of the whole plus 25 gallons was spirits; and a third of the whole minus 5 gallons was water. How many gallons were there of each?

50. At 3 o'clock, A starts upon a journey at the rate of 4 miles an hour, and after 15 minutes B starts from the same place, and follows A at the rate of $4\frac{3}{4}$ miles an hour. When does B overtake A?

51. A fish was caught whose tail weighed 9 pounds; his head weighed as much as his tail and half his body, and his body weighed as much as his head and tail. What did the fish weigh?

52. Find a number such that if $\frac{3}{8}$ of it be subtracted from 20, and $\frac{5}{11}$ of the remainder from $\frac{1}{4}$ of the original number, 12 times the second remainder shall be half the original number.

53. A cistern can be filled in half an hour by a pipe A, and emptied in 20 minutes by another pipe B; after A has been opened 20 minutes, B is also opened for 12 minutes, then A is closed, and B remains open for 5 minutes more, after which there are 13 gallons in the cistern. What was the capacity of the cistern?

54. A father was 24 years old when his eldest son was born; and if both live till the father is twice as old as he now is, the son will then be 8 times as old as now. Find the father's present age.

55. If 19 lbs. of gold weigh 18 lbs. in water, and 10 lbs. of silver weigh 9 lbs. in water, find the quantities of gold and silver in a mass of gold and silver weighing 106 lbs. in air, and 99 lbs. in water.

56. The sum of $1650 is laid out in two investments, by one of which 15 per cent is gained, and by the other 8 per cent is lost; and the amount of the returns is $1725. Find the amount of each investment.

57. How many children are there in a family, if each son has as many brothers as sisters, and each daughter has twice as many brothers as sisters?

116. Interest formulas. In problems of interest, the quantities involved are the *principal, interest, rate, time, and amount.*

Let $p =$ the *number* of dollars in the principal;

$r =$ the rate, or the *ratio* of the *interest per annum* to the principal;

$t =$ the *number* of years in the time;

$i =$ the *number* of dollars in the interest for t years at rate r.

$a =$ the *number* of dollars in the amount, or the sum of the principal and the interest;

then $$i = prt; \tag{1}$$

and $$a = p + prt. \tag{2}$$

Proof. The interest on \$$p$ for one year is \$$pr$; hence i, or the interest on \$$p$ for t years, is \$$prt$; whence (1).

But $a = p + i$; whence (2).

If any three of the four numbers i, p, r, t, or a, p, r, t are given, the fourth can be found by solving equation (1) or (2).

Ex. Find the principal that will amount to \$1584 in 5 years 4 months at 6 per cent.

Here $a = 1584$, $t = 5\frac{1}{3}$, $r = 0.06$. Substituting in (2), we have

$$1584 = p + p\,(0.06)(5\tfrac{1}{3}) = 1.32\,p.$$

$$\therefore p = 1584 \div 1.32 = 1200.$$

Hence the principal is \$1200.

Exercise 40.

1. Solve $i = prt$ for p, r, and t.

2. Solve $a = p + prt$ for p, r, and t.

3. Find the interest on \$4760 for 4 years 6 months at $5\frac{1}{2}$ per cent.

4. Find the amount of $ 3500 for 5 years 4 months at 6 per cent.

5. Find the interest on $ 7240 for 3 years 3 months at 8 per cent.

6. The interest on $ 1250 for 8 months is $ 50. Find the rate per cent.

7. The amount of $ 1050 for 2 years 6 months is $ 1260. Find the rate.

8. The interest on $ 3420 at 6 per cent is $ 649.80. Find the time.

9. A sum of money doubles in 12 years 6 months. Find the rate.

10. Find the principal that will yield $ 262.50 per month at 7 per cent.

11. Find the time in which $ 1350 will amount to $ 1809 at 6 per cent.

12. The interest in 4 years 3 months at 4 per cent is $ 2099.50. Find the principal.

13. Find the time in which a sum of money will double itself at 6 per cent.

14. The interest on $ 1270 for 8 months is $76.20. Find the rate.

15. At 4 per cent how much money is required to yield $ 2500 interest annually?

CHAPTER IX

POWERS, PRODUCTS, QUOTIENTS

117. Certain products and quotients are so frequently required in Algebra that the student should prove and memorize the identities by which they can be written out. In this chapter the most important of these identities are proved, and used in obtaining products and quotients.

In the next chapter the converses of these identities are used for factoring.

118. Power of a power. *The* **n***th power of the* **m***th power of any base is equal to the* **mn***th power of that base; and conversely.*

That is, $$(a^m)^n \equiv a^{mn}.$$

Ex. 1. $(2^3)^2 = 2^3 \times 2^3 = 2^{3+3} = 2^6.$

Ex. 2. $(a^2)^4 \equiv a^2a^2a^2a^2 \equiv a^{2+2+2+2} \equiv a^8.$

Ex. 3. $(a^2)^6 \equiv a^{2\times 6} \equiv a^{12}$; conversely, $a^{2\times 6} \equiv (a^2)^6.$

Proof. $(a^m)^n \equiv a^m a^m \cdots$ to n factors — by notation

$\equiv a^{m+m+\cdots \text{ to } n \text{ summands}}$ — § 76

$\equiv a^{mn}.$

119. Power of a product. *The* **n***th power of the product of two or more factors is equal to the product of the* **n***th powers of those factors; and conversely.*

That is, $$(ab)^n \equiv a^n b^n, \ (abc)^n \equiv a^n b^n c^n.$$

Ex. 1. $(abc)^3 \equiv (abc)(abc)(abc)$ — by notation

$\equiv (aaa)(bbb)(ccc)$ — by laws (A'), (B')

$\equiv a^3b^3c^3.$ — by notation

Proof.

$(ab)^n \equiv (ab)(ab) \cdots$ to n factors — by notation

$\equiv (aa \cdots$ to n factors$)(bb \cdots$ to n factors$)$ — by (A'), (B')

$\equiv a^n b^n.$ — by notation

Similarly for any number of factors.

Ex. 2. $(-a)^5 \equiv (-1)^5 a^5 \equiv -a^5$; conversely, $(-1)^5 a^5 = (-a)^5$.

Ex. 3. $(2x^2y)^4 \equiv 2^4(x^2)^4y^4 \equiv 16x^8y^4$;

conversely, $2^4(x^2)^4y^4 \equiv (2x^2y)^4$.

Ex. 4. $(-5x^2y^3)^3 \equiv (-5)^3(x^2)^3(y^3)^3 \equiv -125x^6y^9$

Exercise 41.

Express as a power or as a product of powers each of the following powers of products:

1. $(-x^2)^4$.
2. $(-x^4)^3$.
3. $(-y^6)^2$.
4. $(-z^7)^3$.
5. $[(-a)^3]^4$.
6. $[(-2)^3]^2$.
7. $(ax^3)^2$.
8. $(-x^3)^4$.
9. $(-ax^2)^5$.
10. $(-by^3)^2$.
11. $(-2ax^3)^2$.
12. $(-a^2x^3)^5$.
13. $(ab^2x^2y)^4$.
14. $(-x^2y \cdot z^3)^6$.
15. $(-2ab^2x^3)^4$.
16. $(-2a^2bc^3)^5$.
17. $(-5a^2x^3y^2)^3$.
18. $(-a^2b^3xy^2)^7$.
19. $(-a)^2, (-a)^3, (-a)^4, (-a)^5, (-a)^6, (-a)^7$.
20. $(-ab)^2, (-ab)^3, (-ab)^4, (-ab)^5, (-ab)^6$.
21. $(-2a^3b^4)^2, (-2a^3b^4)^3, (-2a^3b^4)^4$.
22. As a power of the base 3^2, express $3^4, 3^8, 3^{12}, 3^{20}, 3^{2a}$.
23. As a power of the base x^2, express $x^4, x^6, x^8, x^{10}, x^{12}$.
24. As a power of the base a^3, express $a^6, a^{12}, a^{15}, a^{21}, a^{3b}$.

Express as a power of a product:

25. $6^3 \times 4^3$.
26. $4^5 \times (-3)^5$.
27. $(-a)^5(-b)^5$
28. $(-a)^3b^3$.
29. $(-x)^3y^3(-z)^3$
30. $a^4(x+y)^4$.

120. Square of binomials. By multiplication, we obtain

$$(a+b)^2 \equiv a^2 + 2\,ab + b^2. \qquad (1)$$

That is, *the square of the sum of two numbers is equal to the sum of their squares plus twice their product.*

Ex. 1. $(3\,x + 5\,y)^2 \equiv (3\,x)^2 + (5\,y)^2 + 2\,(3\,x)(5\,y)$ by (1)

$\equiv 9\,x^2 + 25\,y^2 + 30\,xy.$ § 119

Ex. 2. $(2\,x - 3\,y)^2 \equiv [(2\,x) + (-3\,y)]^2$

$\equiv (2\,x)^2 + (-3\,y)^2 + 2\,(2\,x)(-3\,y)$ (2)

$\equiv 4\,x^2 + 9\,y^2 - 12\,xy.$ (3)

Ex. 3. $(a-b)^2 \equiv [a + (-b)]^2$

$\equiv a^2 + (-b)^2 + 2\,a\,(-b)$

$\equiv a^2 - 2\,ab + b^2.$

In the examples of this chapter there are two steps:

First step. The application of an identity.

Second step. The simplification of the result obtained by the first step.

E.g., in example 2, the application of identity (1) gives the result in (2), and the simplification of this result gives (3).

At first the pupil should write out these steps separately; later he should apply the identity mentally, and write only the final result.

Observe the advantage gained in this chapter by regarding a polynomial as a sum.

121. Square of polynomials. If in the identity

$$(a+x)^2 \equiv a^2 + x^2 + 2\,ax \qquad (1)$$

we put $b+y$ for x, we obtain

$$(a+b+y)^2 \equiv a^2 + (b+y)^2 + 2\,a\,(b+y)$$
$$\equiv a^2 + b^2 + y^2 + 2\,ab + 2\,ay + 2\,by. \qquad (2)$$

And so on for a polynomial of any number of terms.

Hence we infer that

The square of any polynomial is equal to the sum of the squares of its several terms, plus the sum of the products of twice each term into each of the terms which follow it.

Ex. $(x^2 + 2y - 3c)^2$

$$\equiv (x^2)^2 + (2y)^2 + (-3c)^2 + 2x^2(2y) + 2x^2(-3c) + 2(2y)(-3c)$$
$$\equiv x^4 + 4y^2 + 9c^2 + 4x^2y - 6cx^2 - 12cy.$$

Exercise 42.

Write the square of each of the following expressions:

1. $2a + x.$
2. $x^2 + y^2.$
3. $3a - 5b.$
4. $3a^3 - b^3x.$
5. $-2a^2 + 5b^2.$
6. $-ax^2 + by^2.$
7. $3abc - 4x^2y.$
8. $-2z^2 - abx^2.$
9. $-4a^2 + 3cz^3.$
10. $a + b + c.$
11. $a + b - c.$
12. $a - b + c.$
13. $a - b - c.$
14. $a + 2b + 4.$
15. $x + 2y + 3z.$
16. $2 + 2x - 3x^2.$
17. $2x^2 - 3x - 2.$
18. $a^3 - 2b^3 + 3c^3.$
19. $3x^2 - 6x - 6.$
20. $x + y + z + v.$
21. $x + y - z - v.$
22. $x - y - z - v.$
23. $x^3 + x^2 - 2x - 2.$
24. $a + 2b - 3c + 4d.$
25. $1 + x - x^2 + x^3.$
26. $3a^2 - x^2 + c^2 - 2y^2.$
27. $4x^2 - 3a - 4c - 3y^2.$
28. $3x^3 - 2a^2 + 4b - y^3.$
29. $8x^2y^3 - 4a^2b^3.$
30. $\frac{1}{2}x^2y^4 + \frac{3}{5}x^4y^2.$
31. $2x^m - 7.$
32. $6x^ny^m - 4x^my^n.$
33. $4a^rb^s - 3a^{r-2}b^{s+1}.$
34. $2(a+1) - 5(b+c).$

122. Product of sum and difference. By multiplication, we have

$$(a+b)(a-b) \equiv a^2 - b^2. \tag{1}$$

That is, *the product of the sum and the difference of the same two numbers is equal to the square of the first number in the difference, minus the square of the second.*

Ex. 1. $(2x^2+5by^3)(2x^2-5by^3) \equiv (2x^2)^2-(5by^3)^2$ by (1)

$\equiv 4x^4-25b^2y^6$ §§ 119, 118

By properly grouping terms, the product of two polynomials can frequently be written as the product of the sum and the difference of the same two numbers.

Ex. 2. $(a+b+c)(a+b-c) \equiv [(a+b)+c][(a+b)-c]$

$\equiv (a+b)^2-c^2$ by (1)

$\equiv a^2+2ab+b^2-c^2.$

Ex. 3. $(a+b-c)(a-b+c) \equiv [a+(b-c)][a-(b-c)]$

$\equiv a^2-(b-c)^2$

$\equiv a^2-b^2+2bc-c^2.$

Exercise 43.

Write each of the following indicated products:

1. $(b+a)(a-b)$.
2. $(5+x)(x-5)$.
3. $(1+3x)(1-3x)$.
4. $(b^2+a^2)(a^2-b^2)$.
5. $(x^2+4y^3)(x^2-4y^3)$.
6. $(3x^2+5y^2)(3x^2-5y^2)$.
7. $(3by+2ax)(2ax-3by)$.
8. $(4cx^2+5b^2y)(4cx^2-5b^2y)$.
9. $(a+b+c)(a-b-c)$.
10. $(1+b-c)(1-b+c)$.
11. $(a-b+c)(a-b-c)$.
12. $(x+3y-2z)(x-3y+2z)$.
13. $(x^2+xy+y^2)(x^2-xy+y^2)$.
14. $(y^2+y+2)(y^2-y+2)$.

15. $(3a + b - 3c)(3a - b + 3c)$.

16. $(a^2 + 3a - 1)(a^2 - 3a - 1)$.

17. $(a^4 - 2a^2 + 1)(a^4 + 2a^2 - 1)$.

18. $(a^2 - b^2 - c^2)(a^2 + b^2 + c^2)$.

19. $(-x^2 - y^2 + 7)(x^2 - y^2 + 7)$.

20. $(ax - by + cz)(ax + by - cz)$.

21. $(3x + 9 - 4y)(3x - 9 + 4y)$.

22. $(1 + 4x + 3y + 2z)(1 + 4x - 3y - 2z)$.

23. $(x + 2y + a - b)(x + 2y - a + b)$.

123. By multiplication, we obtain

$$(x + a)(x + b) \equiv x^2 + (a + b)x + ab.$$

That is, *the product of two binomials having the same first term is equal to the square of the first term, plus the sum of the second terms into the first term, plus the product of the second terms.*

Ex. 1. $$\begin{aligned}(x + 7)(x + 5) &= x^2 + (7 + 5)x + 7 \times 5\\ &= x^2 + 12x + 35.\end{aligned}$$

Ex. 2. $$\begin{aligned}(x - 7)(x - 5) &= [x + (-7)][x + (-5)]\\ &= x^2 + (-7 - 5)x + (-7)(-5)\\ &= x^2 - 12x + 35.\end{aligned}$$

Ex. 3. $$\begin{aligned}(x + 7)(x - 5) &= x^2 + (7 - 5)x + 7(-5)\\ &= x^2 + 2x - 35.\end{aligned}$$

Ex. 4. $$(x - 7)(x + 5) = x^2 - 2x - 35.$$

Exercise 44.

Write each of the following indicated products:

1. $(x + 8)(x + 5)$.
2. $(x - 3)(x + 10)$.
3. $(x + 7)(x - 9)$.
4. $(x - 4)(x + 11)$.
5. $(a + 9)(a - 5)$.
6. $(a - 8)(a + 4)$.

7. $(a-6)(a+13)$.

8. $(x-3a)(x+2a)$.

9. $(a-9b)(a-8b)$.

10. $(3x-2y)(3x+y)$.

$$(3x-2y)(3x+y) \equiv (3x)^2 + (-2y+y)3x + (-2y)y$$
$$\equiv 9x^2 - 3xy - 2y^2.$$

11. $(a-5b)(a+10b)$.

12. $(x^2-6)(x^2+4)$.

13. $(a^2+2x)(a^2-5x)$.

14. $(xy-9)(xy+6)$.

15. $(xy-6ab)(xy+2ab)$.

16. $(ab-5)(ab+7)$.

17. $(xy-7ab)(xy-2ab)$.

18. $(x-4ab)(x+5ab)$.

19. $(xz-9ab)(xz+11ab)$.

20. $(a^n+c)(a^n-b)$.

21. $(x^{n+1}-3)(x^{n+1}-8)$.

22. $(x^{2n-1}-b^2)(x^{2n-1}+c^2)$.

23. $(x^2+4y+4z)(x^2-5y-5z)$.

Regarding $4(y+z)$ and $-5(y+z)$ as the second terms, we have

$$[x^2+4(y+z)][x^2-5(y+z)] \equiv x^4 - (y+z)x^2 - 20(y+z)^2.$$

24. $(x+y+3)(x+y-5)$.

25. $(a+b-7)(a+b-8)$.

26. $(x-y-9)(x-y+8)$.

27. $(a-4-2b)(a+6-2b)$.

124. Cubes of binomials. By multiplication we obtain

$$(\boldsymbol{a}+\boldsymbol{b})^3 \equiv \boldsymbol{a}^3 + 3\boldsymbol{a}^2\boldsymbol{b} + 3\boldsymbol{a}\boldsymbol{b}^2 + \boldsymbol{b}^3. \tag{1}$$

That is, *the cube of the sum of two numbers is the cube of the first, plus three times the square of the first into the second, plus three times the first into the square of the second, plus the cube of the second.*

Ex. 1.

$$(2x+3y)^3 \equiv (2x)^3 + 3(2x)^2(3y) + 3(2x)(3y)^2 + (3y)^3 \quad \text{by (1)}$$
$$\equiv 8x^3 + 36x^2y + 54xy^2 + 27y^3.$$

Ex. 2.

$$(2x-3a)^3 \equiv [(2x)+(-3a)]^3$$
$$\equiv (2x)^3 + 3(2x)^2(-3a) + 3(2x)(-3a)^2 + (-3a)^3$$
$$\equiv 8x^3 - 36x^2a + 54xa^2 - 27a^3.$$

Ex. 3.

$$(a-b)^3 \equiv a^3 - 3a^2b + 3ab^2 - b^3.$$

Observe that when the second term has a negative numeral coefficient, each *even* term in the result contains an *odd* power of this coefficient, and therefore has a negative numeral coefficient.

The same is true in § 120.

125. The operation of raising a number to any required power is called **involution.**

Exercise 45.

Write out the cube of each of the following expressions:

1. $x+1$.	6. $a-2b$.	11. $2ax^2+m^2n$.
2. $2x+a$.	7. $ax+by$.	12. $\frac{1}{2}a^2x^2-\frac{1}{3}b^2y$.
3. $a+3b$.	8. $2a-3bc$.	13. $2x^n+5y^n$.
4. $x-1$.	9. x^2+4a.	14. $3x^my^n+a^2$.
5. $3x-a$.	10. $xy-4ab$.	15. x^nb-3ay^{n+1}.

126. Powers of sums. By multiplication we obtain

$$(a+b)^4 \equiv a^4+4a^3b+6a^2b^2+4ab^3+b^4.$$
$$(a+b)^5 \equiv a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5.$$
$$(a+b)^6 \equiv a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6.$$

The expression obtained by performing the indicated operation in $(a+b)^n$ is called the **expansion** of $(a+b)^n$. Thus, the second member of each of the above identities is the *expansion* of its first member.

By inspection we discover in each of these expansions the following laws of exponents and coefficients:

(i) *The exponent of* **a** *in the first term is equal to the exponent of the binomial, and it decreases by* 1 *from term to term.*

(ii) *The exponent of* **b** *in the second term is* 1, *and increases by* 1 *from term to term.*

(iii) *The coefficient of the first term is* 1, *and that of the second term is the exponent of the binomial.*

(iv) *If in any term the coefficient is multiplied by* **a's** *exponent, and this product is divided by* **b's** *exponent plus* 1, *the result is the coefficient of the next term.*

E.g., in the expansion of $(a+b)^5$, from the second term $5\,a^4b$, by (iv), we obtain $5 \times 4 \div 2$, or 10, which is the coefficient of the third term. From the third term $10\,a^3b^2$, by (iv), we obtain $10 \times 3 \div 3$, or 10, which is the coefficient of the fourth term; and so on. It can be proved that the above laws hold for any power of a binomial.

In the expansion of $(a+b)^4$ there are 5 terms, each of the fourth degree in a and b, and the first two coefficients are repeated in inverse order after the third term. In the expansion of $(a+b)^5$ there are 6 terms, each of the fifth degree in a and b, and the first three coefficients are repeated in inverse order after the third term.

Observe that in each of the above expansions:

The sum of the exponents of a and b in any term is equal to the exponent of the binomial.

The number of terms is equal to the exponent of the binomial plus 1.

The coefficients are repeated in the inverse order after passing the middle term or half the number of terms, so that the coefficients of the last half of the expansion can be written out from the first half.

Each expansion is homogeneous in a and b.

Ex. 1. $(2\,x+3\,b)^4$

$$\equiv (2\,x)^4+4(2\,x)^3(3\,b)+6(2\,x)^2(3\,b)^2+4(2\,x)(3\,b)^3+(3\,b)^4$$
$$\equiv 16\,x^4+96\,x^3b+216\,x^2b^2+216\,xb^2+81\,b^4.$$

Ex. 2. $(2\,x-a)^5$

$$\equiv (2\,x)^5+5(2\,x)^4(-a)+10(2\,x)^3(-a)^2+10(2\,x)^2(-a)^3$$
$$+5(2\,x)(-a)^4+(-a)^5$$
$$\equiv 32\,x^5-80\,x^4a+80\,x^3a^2-40\,x^2a^3+10\,xa^4-a^5.$$

Ex. 3. $(a-b)^6$

$$\equiv a^6-6\,a^5b+15\,a^4b^2-20\,a^3b^3+15\,a^2b^4-6\,ab^5+b^6.$$

Observe that when the second term in the binomial has a negative numeral coefficient, each *even* term in the expansion contains an *odd* power of this coefficient, and therefore has a negative numeral coefficient. Thus the expansion of $(x-y)^6$ differs from that of $(x+y)^6$ only in the signs before the even terms.

Exercise 46.

Expand each of the following powers:

1. $(2x+1)^4$.
2. $(2-3y)^4$.
3. $(x^2+xy)^4$.
4. $(2x+a)^5$.
5. $(2a-3)^5$.
6. $(a^2+bc)^5$.
7. $(2x+3)^6$.
8. $(3a-2b)^6$.
9. $(2m^2-an^2)^6$.
10. $(a^n-2b^2)^4$.
11. $(a^nb-x^my)^5$.
12. $(2a-c+x^2)^3$.

Since any polynomial can be written as a binomial, the laws in § 126 can be used to expand a power of a polynomial.

E.g., we can write,

$$\begin{aligned}(2a-c+x^2)^3 &\equiv [(2a-c)+x^2]^3\\ &\equiv (2a-c)^3+3(2a-c)^2(x^2)+3(2a-c)(x^2)^2+(x^2)^3\\ &\equiv 8a^3-12a^2c+6ac^2-c^3+12a^2x^2-12acx^2\\ &\qquad +3c^2x^2+6ax^4-3cx^4+x^6.\end{aligned}$$

13. $(x^2+x+1)^3$.
14. $(x^2-x+2)^3$.
15. $(3x^2-5x+1)^3$.
16. $(2x-3a+b)^3$.

127. Two powers are said to be **like**, when their *exponents* are equal; and **unlike**, when their *exponents* are unequal.

E.g., a^2, x^2 are *like* powers; a^2, a^3, a^4 are *unlike* powers.

128. By 6 of § 6, like powers of equal numbers are equal. Hence, *like powers of identical expressions are identical.*

129. By division we obtain,

$$(a^3-b^3)\div(a-b)\equiv a^2+ab+b^2;$$
$$(a^4-b^4)\div(a-b)\equiv a^3+a^2b+ab^2+b^3;$$
$$(a^5-b^5)\div(a-b)\equiv a^4+a^3b+a^2b^2+ab^3+b^4.$$

From the above identities we infer the following theorem:

The **difference** *of the* **like** *powers of any two numbers, as* $\boldsymbol{a}$ *and* $\boldsymbol{b}$*, is exactly divisible by the* **difference** *of the numbers, taken in the same order;* the laws of exponents and coefficients in the quotients being as follows:

(i) *The exponent of* $\boldsymbol{a}$ *in the first term is* 1 *less than the exponent of* $\boldsymbol{a}$ *in the dividend, and it decreases by* 1 *from term to term.*

(ii) *The exponent of* $\boldsymbol{b}$ *is* 1 *in the second term and increases by* 1 *from term to term.*

(iii) *The numeral coefficient of each term is* $+1$. *The number of terms is equal to the exponent of* $\boldsymbol{a}$ *in the dividend.*

Or stated in symbols the theorem is

$$\frac{\boldsymbol{a}^n - \boldsymbol{b}^n}{\boldsymbol{a} - \boldsymbol{b}} \equiv \boldsymbol{a}^{n-1} + \boldsymbol{a}^{n-2}\boldsymbol{b} + \boldsymbol{a}^{n-3}\boldsymbol{b}^2 + \cdots + \boldsymbol{a}\boldsymbol{b}^{n-2} + \boldsymbol{b}^{n-1}, \quad (1)$$

where n is any positive integer.

Proof. Multiplying the second member of (1) by the divisor $a - b$, we obtain the dividend $a^n - b^n$.
Hence (1) is an identity (§ 83).

Ex. 1. $\dfrac{27\,a^3 - b^3}{3\,a - b} \equiv \dfrac{(3\,a)^3 - b^3}{(3\,a) - b}$

$$\equiv (3\,a)^2 + (3\,a)b + b^2 \equiv 9\,a^2 + 3\,ab + b^2.$$

Ex. 2.

$$\frac{32\,a^5 - 243\,x^5}{(2\,a) - (3\,x)} \equiv (2\,a)^4 + (2\,a)^3(3\,x) + (2\,a)^2(3\,x)^2 + 2\,a(3\,x)^3 + (3\,x)^4$$
$$\equiv 16\,a^4 + 24\,a^3x + 36\,a^2x^2 + 54\,ax^3 + 81\,x^4.$$

Observing that each term in the dividend is the fifth power of the corresponding term in the divisor, we write the quotient by taking the proper powers of $2\,a$ and $3\,x$.

The quotient is homogeneous when the dividend is homogeneous.

Exercise 47.

Write each of the following indicated quotients:

1. $\frac{9x^4 - 4y^2}{3x^2 - 2y}$.

2. $\frac{81a^4 - 16b^6}{9a^2 - 4b^3}$.

3. $\frac{x^3y^3 - a^3}{xy - a}$.

4. $\frac{64a^3 - 8x^3}{4a - 2x}$.

5. $\frac{1 - 729y^3}{1 - 9y}$.

6. $\frac{a^4 - 16}{a - 2}$.

7. $\frac{81a^4 - 1}{3a - 1}$.

8. $\frac{x^5 - 32}{x - 2}$.

9. $\frac{32x^5 - 1}{2x - 1}$.

10. $\frac{243a^5 - 32b^5}{3a - 2b}$.

11. $\frac{8a^6x^6 - 343b^3}{2a^2x^2 - 7b}$.

12. $\frac{64a^9b^3 - 8x^6y^3}{4a^3b - 2x^2y}$.

13. $\frac{x^{2n+2} - 4}{x^{n+1} - 2}$.

14. $\frac{x^{3n+3} - y^{6n}}{x^{n+1} - y^{2n}}$.

15. $\frac{x^{4n+4} - y^{8n}}{x^{n+1} - y^{2n}}$.

130. By division, we obtain:

$$\text{(i)} \quad \begin{cases} (a^2 - b^2) \div (a + b) \equiv a - b; \\ (a^4 - b^4) \div (a + b) \equiv a^3 - a^2b + ab^2 - b^3. \end{cases}$$

$$\text{(ii)} \quad \begin{cases} (a^3 + b^3) \div (a + b) \equiv a^2 - ab + b^2; \\ (a^5 + b^5) \div (a + b) \equiv a^4 - a^3b + a^2b^2 - ab^3 + b^4. \end{cases}$$

From the identities in (i) and (ii) we infer the two following theorems:

(i) *The* **difference** *of the like* **even** *powers of two numbers is exactly divisible by the* **sum** *of the numbers.*

(ii) *The* **sum** *of the like* **odd** *powers of two numbers is exactly divisible by the* **sum** *of the numbers.*

In each quotient, the laws of exponents and the number of terms are the same as in § 129.

The numeral coefficient of any odd term is $+1$, *and that of any even term is* -1.

Or stated in symbols, when n is even, the theorem is

$$\frac{a^n - b^n}{a + b} \equiv a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \cdots + ab^{n-2} - b^{n-1}, \quad (1)$$

and, when n is odd, the theorem is

$$\frac{a^n + b^n}{a + b} \equiv a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \cdots - ab^{n-2} + b^{n-1}. \quad (2)$$

Proof. Multiplying the second member of (1) or (2) by the divisor $a + b$, we obtain the dividend $a^n - b^n$ or $a^n + b^n$. Hence (1) and (2) are identities (§ 83).

Ex. 1. $$\frac{16\,x^4 - 81\,y^4}{2\,x + 3\,y} \equiv \frac{(2\,x)^4 - (3\,y)^4}{2\,x + 3\,y}$$
$$\equiv (2\,x)^3 - (2\,x)^2(3\,y) + (2\,x)(3\,y)^2 - (3\,y)^3$$
$$\equiv 8\,x^3 - 12\,x^2y + 18\,xy^2 - 27\,y^3.$$

Ex. 2. $$\frac{8\,a^3b^6 + 216\,c^3x^9}{2\,ab^2 + 6\,cx^3} \equiv (2\,ab^2)^2 - (2\,ab^2)(6\,cx^3) + (6\,cx^3)^2$$
$$\equiv 4\,a^2b^4 - 12\,ab^2cx^3 + 36\,c^2x^6.$$

Exercise 48.

Write each of the following indicated quotients:

1. $\dfrac{1 - a^2b^2}{1 + ab}$.
2. $\dfrac{4\,a^2x^2 - 9\,y^4}{2\,ax + 3\,y^2}$.
3. $\dfrac{9\,x^6 - 16\,y^2}{3\,x^3 + 4\,y}$.
4. $\dfrac{a^4b^8 - x^8y^{12}}{ab^2 + x^2y^3}$.
5. $\dfrac{x^4y^{12} - 16\,m^8}{xy^3 + 2\,m^2}$.
6. $\dfrac{x^4y^8 - 1}{xy^2 + 1}$.
7. $\dfrac{x^3 + 1}{x + 1}$.
8. $\dfrac{x^3 + 8}{x + 2}$.
9. $\dfrac{1 + 8\,a^3}{1 + 2\,a}$.
10. $\dfrac{x^3y^3 + 216\,z^3}{xy + 6\,z}$.
11. $\dfrac{a^6b^6 + 8\,c^3}{a^2b^2 + 2\,c}$.
12. $\dfrac{x^5y^5 + 1}{xy + 1}$.
13. $\dfrac{a^5 + 32}{a + 2}$.
14. $\dfrac{a^5b^5 + 243}{ab + 3}$.
15. $\dfrac{729 + 8\,b^6}{9 + 2\,b^2}$.
16. $\dfrac{a^{10}b^{10} + 32\,x^5}{a^2b^2 + 2\,x}$.
17. $\dfrac{16\,x^4y^4 - 256\,a^8}{2\,xy + 4\,a^2}$.
18. $\dfrac{a^6b^{12} + x^9y^{18}}{a^2b^4 + x^3y^6}$.
19. Make a list of the identities proved in this chapter.

131. The remainder theorem. *The result obtained by substituting* a *for* x *in any integral expression in* x *is the same as the remainder arising from dividing the expression by* $x - a$.

E.g., dividing $2x^3 - x^2 - 6$ by $x - 2$ until the remainder does not contain x, we obtain the remainder 6, and 6 is what $2x^3 - x^2 - 6$ equals when $x = 2$.

Again, dividing $x^3 + a^3$ by $x - a$, we obtain the remainder $2a^3$, and $2a^3$ is what $x^3 + a^3$ equals when $x = a$.

Proof. Let P denote any integral expression in x.

Divide P by $x - a$ until the remainder does not contain x.

Let Q denote the quotient and R the remainder; then

$$P \equiv Q(x - a) + R. \tag{1}$$

Let $P]_a$ (read 'P for $x = a$') denote the value of P when a is substituted for x.

Put a for x in (1); then, observing that $Q]_a(a - a)$ is zero, and that R does not contain x, we have

$$P]_a \equiv R. \tag{2}$$

132. The factor theorem. *If any integral expression in* x *becomes zero when* a *is substituted for* x, *the expression is exactly divisible by* $x - a$.

Proof. From $P]_a \equiv R$ in § 131, it follows that if $P]_a = 0$, the remainder is zero, and the division is exact.

Ex. 1. The expression $x^5 - a^5$ becomes zero when a is put for x; hence $x^5 - a^5$ is exactly divisible by $x - a$.

Ex. 2. The expression $x^7 + y^7$ becomes zero when $-y$ is put for x; hence $x^7 + y^7$ is exactly divisible by $x - (-y)$, or $x + y$.

Ex. 3. The expression $a^n - b^n$ becomes zero when b is put for a; hence $a^n - b^n$ is exactly divisible by $a - b$.

Ex. 4. When n is odd, $a^n + b^n$ becomes zero when $-b$ is put for a; hence $a^n + b^n$ is exactly divisible by $a - (-b)$, or $a + b$.

Exercise 49.

By § 132, prove that each of the following dividends is exactly divisible by the corresponding divisor:

1. $(x^2 - x - 6) \div (x - 3)$.
2. $(x^2 - 2x - 15) \div (x - 5)$.
3. $(x^3 - 14x - 8) \div (x - 4)$.
4. $(x^3 - 3x^2 + 4) \div (x + 1)$.
5. $(2x^4 - 3x^2 - 4x + 5) \div (x - 1)$.
6. $(x^4 - 2a^2x^2 + 4a^3x - 3a^4) \div (x - a)$.
7. $(x^4 - 3a^2x^2 - 7a^3x - 5a^4) \div (x + a)$.

When n is odd, by §§ 131 and 132, prove:

8. $x^n + a^n$ is exactly divisible by $x + a$, but not by $x - a$.

9. $x^n - a^n$ is exactly divisible by $x - a$, but not by $x + a$.

When n is even, prove:

10. $x^n - a^n$ is exactly divisible by both $x + a$ and $x - a$.

11. $x^n + a^n$ is not exactly divisible by either $x + a$ or $x - a$.

133. The following examples illustrate how the formulas in § 129 or § 130 often aid in writing out the partial quotient and the remainder, when a division is not exact.

Ex. 1. Divide $a^2 + b^2$ by $a - b$.

Adding to the dividend zero in the form $-b^2 + b^2$, we have

$$\frac{a^2 + b^2}{a - b} \equiv \frac{a^2 - b^2 + 2b^2}{a - b} \equiv a + b + \frac{2b^2}{a - b}.$$

Ex. 2. Divide $a^3 + 1$ by $a - 1$.

Adding to the dividend zero in the form $-1 + 1$, we have

$$\frac{a^3 + 1}{a - 1} \equiv \frac{a^3 - 1 + 2}{a - 1} \equiv a^2 + a + 1 + \frac{2}{a - 1}.$$

Ex. 3. $$\frac{x^4 + 1}{x + 1} \equiv \frac{x^4 - 1 + 2}{x + 1} \equiv x^3 - x^2 + x - 1 + \frac{2}{x + 1}.$$

Ex. 4. $$\frac{a^3 + 3}{a - 2} \equiv \frac{a^3 - 8 + 11}{a - 2} \equiv a^3 + 2a + 4 + \frac{11}{a - 2}.$$

Exercise 50.

Write the partial quotients and the remainders:

1. $\dfrac{x^2+4}{x-2}$.
2. $\dfrac{x^2+5}{x+1}$.
3. $\dfrac{x^3+3}{x-1}$.
4. $\dfrac{x^3+a^3}{x-a}$.
5. $\dfrac{x^3-3}{x-2}$.
6. $\dfrac{a^3x^3+y^6}{ax-y^2}$.
7. $\dfrac{x^4+a^4}{x-a}$.
8. $\dfrac{x^4+a^4}{x+a}$.
9. $\dfrac{x^4+1}{x+1}$.
10. $\dfrac{x^5+a^5}{x-a}$.
11. $\dfrac{x^5-a^5}{x+a}$.
12. $\dfrac{x^5+6}{x-2}$.

Simplify each of the following expressions:

13. $(1+a)^2-(1+a)(1-a)$.
14. $(1-x^2)^2+(1+x^2)(1-x^2)$.
15. $(a+b)^2(a-b)^2$.
16. $(2y-3a)^2(2y+3a)^2$.
17. $(x-a)(x+a)(x^2+a^2)(x^4+a^4)$.
18. $(x^2-a^2)(x^8+a^8)(x^4+a^4)(x^2+a^2)$.
19. $(x^2-x+1)(x^2+x+1)(x^4-x^2+1)$.
20. $(x+y+z)(x+z-y)(y+z-x)(x+y-z)$.

Write each of the following indicated quotients:

21. $\dfrac{a^6b^{12}-x^6y^{12}}{ab^2-x^2y^3}$.
22. $\dfrac{64a^{12}-729x^{18}}{2a^2+3x^3}$.
23. $\dfrac{x^7y^7+128}{xy+2}$.
24. $\dfrac{a^{14}x^{7n}+b^{14m}}{a^2x^n+b^{2m}}$.
25. $\dfrac{a^{6m}-64x^{6n+6}}{a^m-2x^{n+1}}$.
26. $\dfrac{b^{12m}-729y^{6n+6}}{b^{2m}-3y^{n+1}}$.
27. $\dfrac{a^{5m}b^{10m}-32x^{5n}y^{5n+5}}{a^mb^{2m}-2x^ny^{n+1}}$.
28. $\dfrac{x^{5m}+(a+b)^{5m+5}}{x^m+(a+b)^{m+1}}$.

CHAPTER X

FACTORS OF INTEGRAL LITERAL EXPRESSIONS

134. The problem of multiplication is 'given two or more factors, to find their product.' The converse problem, 'given a product, to find its factors,' is the problem of *factoring*. Reread §§ 33, 117.

Certain forms of products which frequently occur are called type-forms, as $a^2 + 2\,ab + b^2$ or $a^2 - b^2$.

135. Any *monomial* is readily resolved into its factors.

E.g., the factors of $5\,x\,(a + y)$ are 5, x, and $a + y$.

The factors of xy are x and y or $-x$ and $-y$; but we usually use the factors x and y because of their simpler form, unless there is some special reason for using $-x$ and $-y$.

Again, the factors of x^2 are x and x or $-x$ and $-x$; that is, x^2 is the square of x or $-x$.

136. The converse of the distributive law is

$$\boldsymbol{ax + bx + cx + \cdots \equiv (a + b + c + \cdots)x.} \qquad (1)$$

Hence, *any factor which is common to all the terms of a polynomial is a factor of the polynomial.*

Ex. 1. Factor $3\,ax^2 + 6\,a^2x - 9\,a^3x^3$.

Here $3\,ax$ is seen to be a factor of each term; hence

$$\begin{aligned} 3\,ax^2 + 6\,a^2x - 9\,a^3x^3 &\equiv x(3\,ax) + 2\;a(3\,ax) + (-3\,a^2x^2)\,(3\,ax) \\ &\equiv (x + 2\,a - 3\,a^2x^2)\,3\,ax. \end{aligned}$$

Hence the required factors are 3, a, x, and $x + 2\,a - 3\,a^2x^2$.

In identity (1) the letters x, a, b, c, $\cdots$ can stand for any binomial or polynomial.

Ex. 2. Factor $x(a-3b)-2y(a-3b)$.

The binomial $a-3b$ is a factor of each term; hence

$$x(a-3b)-2y(a-3b) \equiv (x-2y)(a-3b).$$

Ex. 3. $$y-x-2a(y-x) \equiv 1(y-x)-2a(y-x)$$
$$\equiv (1-2a)(y-x).$$

Ex. 4. Factor $(y-x)(a^2+b)-2(y-x)(a^2-b)$.

The expression $$\equiv (a^2+b)(y-x)-2(a^2-b)(y-x)$$
$$\equiv [a^2+b-2(a^2-b)](y-x)$$
$$\equiv (3b-a^2)(y-x).$$

Ex. 5. $$a^2(n-x)-b^2(x-n) \equiv a^2(n-x)+b^2(n-x)$$
$$\equiv (a^2+b^2)(n-x).$$

Exercise 51.

Factor each of the following expressions:

1. $3x+3$.
2. x^2+5x.
3. $ab+bc$.
4. $4a^2-6a^2b$.
5. $2ax+3x^2$.
6. $7a^3-21a^2b$.
7. $x^3-5x^2y+20x^2y^2$.
8. $5ax^2-10a^2x-5a^2x^2$.
9. $38a^3b^3-57a^4b^2$.
10. $3a^3b-6a^2b^2+9a^2b^3$.
11. $15a^3b-10a^3c+5a^3d$.
12. $8a^2x-4a^2y-12a^2b^2$.
13. $2a^ny^n+6a^{n+2}y^{n+1}$.
   Ans. $2,\ a^n,\ y^n,\ 1+3a^2y$.
14. $ax^{m+2}y^{n+1}+bx^{m+3}y^{n+2}$.
   Ans. $x^{m+2},\ y^{n+1},\ a+bxy$.
15. $6y^{m+2}-3y^m$.
16. $8x^{2n}-4x^n$.
17. $7x^{m+1}-14x$.
18. $x(a+1)-y(a+1)$.
19. $y(x-a)-x+a$.
20. $y(x-a)-(a-x)$.
21. $4(x+1)^2-6(x+1)$.
22. $x(y-b)^2-c(b-y)$.

137. Trinomials of the type-form $a^2 + 2ab + b^2$.

The converse of the identity in § 120 is

$$a^2 + 2ab + b^2 \equiv (a + b)^2. \tag{1}$$

That is, *a trinomial, two of whose terms are the squares of two numbers respectively, and the remaining term is twice the product of these numbers, is equal to the square of the sum of these numbers.*

Ex. 1. Factor $9x^2 + 24x + 16$.

$9x^2$ is the square of $3x$, 16 is the square of 4, and

$$24x \equiv 2 \cdot 3x \cdot 4.$$

$$\therefore\ 9x^2 + 24x + 16 \equiv (3x + 4)^2. \tag{1}$$

Or, $9x^2$ is the square of $-3x$, 16 is the square of -4, and

$$24x \equiv 2(-3x)(-4).$$

$$\therefore\ 9x^2 + 24x + 16 \equiv (-3x - 4)^2. \tag{2}$$

The factors in either (1) or (2) are correct, but unless there is some reason to the contrary we usually take the simpler factors given in (1).

Ex. 2. Factor $36a^4 + b^4 - 12a^2b^2$.

$36a^4$ is the square of $6a^2$ or $-6a^2$, and b^4 is the square of b^2 or $-b^2$.

To obtain the term $-12a^2b^2$ we must take either $6a^2$ and $-b^2$ or $-6a^2$ and b^2; that is,

$$-12a^2b^2 \equiv 2 \cdot 6a^2(-b^2),\ \text{or}\ 2(-6a^2)b^2.$$

$$\therefore\ 36a^4 + b^4 - 12a^2b^2 \equiv (6a^2 - b^2)^2,\ \text{or}\ (-6a^2 + b^2)^2.$$

Any polynomial which is to be factored should be *first examined for any factors common to all its terms.*

Ex. 3. $-3a^5 + 3a^4b^3 - 75a^3b^3 \equiv -3a^3(a^2 - 10ax^3 + 25b^6)$
$$\equiv -3a^3(a - 5b^3)^2.$$

In identity (1), a and b can denote any binomial or polynomial.

Ex. 4. $(x - 2y)^2 + 2(x - 2y)(3y - 2x) + (3y - 2x)^2$
$$\equiv [(x - 2y) + (3y - 2x)]^2 \equiv (y - x)^2.$$

Exercise 52.

Factor each of the following expressions:

1. $a^2 + 6a + 9$.
2. $x^2 + 12x + 36$.
3. $x^2 + 25 + 10x$.
4. $x^2 + 121 - 22x$.
5. $a^2 + 49 - 14a$.
6. $a^2 + 25 - 10a$.
7. $1 - 8x + 16x^2$.
8. $4a^2 + 9b^2 - 12ab$.
9. $9a^4 + 24a^2b^2 + 16b^4$.
10. $x^2 + \frac{1}{4}y^2 + xy$.
11. $5a^4 - 10a^2b + 5b^2$.
12. $a^3 - 6a^2b + 9ab^2$.
13. $4x^2y^2 - x^4 - 4y^4$.
14. $8x^2 - 4x^4 - 4$.
15. $x^2y^2 + x^3y + \frac{1}{4}xy^3$.
16. $4a^2x^2 + 4abxy + b^2y^2$.
17. $9a^2 + 25b^2 - 30ab$.
18. $25a^4x^2 - 30a^2b^2x + 9b^4$.
19. $25a^4 + 25b^2 - 50a^2b$.
20. $a^2 + 25b^6 - 10ab^3$.
21. $\frac{4}{9}a^4 + \frac{16}{25}b^2 - \frac{16}{15}a^2b$.
22. $4xy^3 - 4x^2y^2 + x^3y$.
23. $(a+b)^2 + 2(a+b) + 1$.
24. $(2x-a)^2 - 8(a-2x) + 16$.

$$\begin{aligned}(2x-a)^2 - 8(a-2x) + 16 &\equiv (2x-a)^2 + 8(2x-a) + 16.\\ &\equiv (2x-a+4)^2.\end{aligned}$$

25. $(x^2 + 2xy + y^2)a + (x+y)b^2$.

$$\begin{aligned}(x^2 + 2xy + y^2)a + (x+y)b^2 &\equiv (x+y)^2a + (x+y)b^2.\\ &\equiv (ax + ay + b^2)(x+y).\end{aligned}$$

26. $x^2(x+2) + 2(x+2)^2 + 2x(x+2)$.
27. $m^2 + 2mn + n^2 - p(m+n)$.
28. $a(b-c) - (b^2 - 2bc + c^2)$.
29. $x^{2n} + 2x^ny^m + y^{2m}$.
30. $36x^{n+2} - 48x^{n+1} + 16x^n$.

138. A perfect square which contains only two different powers of some one letter can often be reduced to the *type-form* $a^2 + 2ab + b^2$ by first writing the polynomial in descending powers of that letter.

Ex. 1. Factor $x^2 + y^2 + z^2 + 2\,xy - 2\,xz - 2\,yz.$

The expression contains only two different powers of x; hence, we arrange the expression in descending powers of x, as follows:

$$\begin{aligned}\text{The expression} &\equiv x^2 + 2\,x(y - z) + (y^2 + z^2 - 2\,yz)\\ &\equiv x^2 + 2\,x(y - z) + (y - z)^2\\ &\equiv (x + y - z)^2.\end{aligned}$$

We could have arranged this expression in descending powers of y or z.

Ex. 2. Factor $a^4 + 4\,b^4 + 9\,c^4 + 4\,a^2b^2 - 6\,a^2c^2 - 12\,b^2c^2.$

Arranging the expression in descending powers of a, we have

$$\begin{aligned}\text{The expression} &\equiv a^4 + 2\,a^2(2\,b^2 - 3\,c^2) + (4\,b^4 + 9\,c^4 - 12\,b^2c^2)\\ &\equiv a^4 + 2\,a^2(2\,b^2 - 3\,c^2) + (2\,b^2 - 3\,c^2)^2\\ &\equiv (a^2 + 2\,b^2 - 3\,c^2)^2.\end{aligned}$$

Ex. 3. Factor

$$a^6 - 2\,a^5 + 3\,a^4 + 2\,a^3(b - 1) + a^2(1 - 2\,b) + 2\,ab + b^2.$$

The expression contains only two different powers of b; hence, we arrange it in descending powers of b, as follows:

$$b^2 + 2\,b(a^3 - a^2 + a) + (a^6 - 2\,a^5 + 3\,a^4 - 2\,a^3 + a^2).$$

This expression is a perfect square, if its last term is the square of $a^3 - a^2 + a$. By § 121, we have

$$(a^3 - a^2 + a)^2 \equiv a^6 - 2\,a^5 + 3\,a^4 - 2\,a^3 + a^2.$$

Hence the given expression is identical with

$$b^2 + 2\,b(a^3 - a^2 + a) + (a^3 - a^2 + a)^2,$$

or
$$(b + a^3 - a^2 + a)^2.$$

Exercise 53.

Factor each of the following expressions:

1. $c^2 - 6\,c(a + b) + 9(a + b)^2.$
2. $a^2 + b^2 + 4\,c^2 + 2\,ab + 4\,ac + 4\,bc.$
3. $4\,a^2 + b^2 + 9\,c^2 + 6\,bc - 12\,ac - 4\,ab.$
4. $4\,a^4 + b^4 + c^4 - 2\,b^2c^2 - 4\,a^2c^2 + 4\,a^2b^2.$

5. $x^2 + 4y^2 + 9z^2 + 4xy + 6xz + 12yz.$

6. $25a^4 + 9b^4 + 4c^4 - 12b^2c^2 + 20c^2a^2 - 30a^2b^2.$

7. $6acx^5 + 4b^2x^4 + a^2x^{10} + 9c^2 - 12bcx^2 - 4abx^7.$

8. $-6b^2c^2 + 9c^4 + b^4 - 12c^2a^2 + 4a^4 + 4a^2b^2.$

9. $6ab^2c - 4a^2bc + a^2b^2 + 4a^2c^2 + 9b^2c^2 - 12abc^2.$

NOTE. The products in exercise 53 can be factored by using the converse of § 121.

139. Trinomials of the type-form $x^2 + px + q$.

The converse of the identity in § 123 is

$$x^2 + (a + b)x + ab \equiv (x + a)(x + b). \qquad (1)$$

Any trinomial in the form $x^2 + px + q$ can be written in the form $x^2 + (a + b)x + ab$ and factored by (1), when we know the two factors of q whose sum is p.

The two factors of q whose sum is p can often be found by inspection as below:

Ex. 1. Factor $x^2 + 7x + 12$.

Here $p = 7$ and $q = 12$.

The two factors of $+12$ are both $+$, or both $-$; hence, as their sum is $+7$, both are $+$. The pairs of positive whole numbers whose product is 12, are 12 and 1, 6 and 2, 4 and 3; since $4 + 3 = 7$, 3 and 4 are the two factors of 12 whose sum is 7.

$$\begin{aligned} \therefore x^2 + 7x + 12 &\equiv x^2 + (3 + 4)x + 3 \times 4 \\ &\equiv (x + 3)(x + 4). \qquad \text{by (1)} \end{aligned}$$

Ex. 2. Factor $x^2 - 9x + 20$.

The two factors of $+20$ are both $+$ or both $-$; hence, as their sum is -9, both are $-$. The pairs of negative whole numbers whose product is 20 are -20 and -1, -10 and -2, -5 and -4; since $(-5) + (-4) = -9$, -5 and -4 are the two factors of 20 whose sum is -9.

$$\begin{aligned} \therefore x^2 - 9x + 20 &\equiv x^2 + (-5 - 4)x + (-5)\cdot(-4) \\ &\equiv (x - 5)(x - 4). \qquad \text{by (1)} \end{aligned}$$

Ex. 3. Factor $x^2 + 6x - 27$.

The two factors of -27 are opposite numbers; hence, as their sum is $+6$, the positive factor is arithmetically the larger. The pairs of whole numbers whose product is -27, the larger arithmetically being $+$, are 27 and -1, 9 and -3; since $9 + (-3) = 6$, 9 and -3 are the required factors.

$$\therefore\ x^2 + 6x - 27 \equiv x^2 + (9-3)x + 9 \cdot (-3)$$
$$\equiv (x+9)(x-3). \qquad \text{by (1)}$$

Ex. 4. Factor $a^2x^2 - 5ax - 84$.

The two factors of -84 are opposite numbers; hence, as their sum is -5, the negative factor is arithmetically the larger. The pairs of whole numbers whose product is -84, the larger arithmetically being $-$, are -84 and $+1$, -42 and $+2$, -28 and $+3$, -21 and $+4$, -14 and $+6$, -12 and 7; since $-12 + 7 = 5$, -12 and $+7$ are the required factors.

$$\therefore\ (ax)^2 - 5(ax) - 84 \equiv (ax - 12)(ax + 7).$$

Ex. 5. $9x^2 - 12x - 77 \equiv (3x)^2 - 4(3x) - 77$
$$\equiv (3x - 11)(3x + 7).$$

Ex. 6. Factor $x^2 - 32xy - 105y^2$.

The two factors of $-105y^2$ whose sum is $-32y$ are $3y$ and $-35y$.

$$\therefore\ x^2 - 32xy - 105y^2 \equiv (x + 3y)(x - 35y).$$

Ex. 7. $4a - a^2 + 21 \equiv -(a^2 - 4a - 21)$
$$\equiv -(a-7)(a+3)$$
$$\equiv (7-a)(a+3).$$

Exercise 54.

Factor each of the following expressions:

1. $x^2 + 4x + 3$.
2. $x^2 - 4x + 3$.
3. $x^2 + 9x + 20$.
4. $x^2 - 11x + 18$.
5. $x^2 - 8x + 15$.
6. $x^2 + 2x - 3$.
7. $x^2 + x - 6$.
8. $x^2 + 4x - 5$.
9. $x^2 + 2x - 35$.
10. $x^2 - 3x - 10$.

11. $x - x^2 + 6$.
12. $x^2 + 5x + 14$.
13. $x^2 + 18x + 72$.
14. $x - x^2 + 132$.
15. $x^2 - 5x - 84$.
16. $x^2 + 5x - 150$.
17. $x^2 - 25x + 150$.
18. $x^2 + 11x - 180$.
19. $x - x^2 + 156$.
20. $x^2 - 31x + 240$.
21. $x^2 - 34x + 288$.
22. $x^2 - 35x - 200$.
23. $x^2 - 17x - 200$.
24. $a^2x^2 - 21ax + 108$.
25. $a^2x^2 - 21ax + 80$.
26. $a^2x^2 + 21ax + 90$.
27. $a^2x^2 - 19ax + 78$.
28. $a^2x^2 + 30ax + 225$.
29. $a^2x^2 + 54ax + 729$.
30. $a^2x^2 - 38ax + 361$.
31. $x^2y^2 - 5xy - 24$.
32. $4x^2 + 12x - 55$.
33. $9x^2 + 6x - 35$.
34. $16x^2 + 8x - 15$.
35. $4x^2 - 12x - 91$.
36. $x^2 - 20xy - 96y^2$.
37. $x^2 - 26xy + 169y^2$.
38. $x^2 - 23xy + 132y^2$.
39. $4x^2 + 20xy + 21y^2$.
40. $9x^2 - 39xy + 22y^2$.
41. $x^2 + 43xy + 390y^2$.
42. $a^2 - 20abx + 75b^2x^2$.
43. $a^2 - 29ab + 54b^2$.
44. $130 + 31xy + x^2y^2$.
45. $a^2 + 12abx - 28b^2x^2$.
46. $x^4 + 13a^2x^2 - 300a^4$.
47. $x^4 - a^2x^2 - 462a^4$.
48. $x^4 - a^2x^2 - 132a^4$.
49. $143 - 24xa + x^2a^2$.
50. $216 + 35x + x^2$.
51. $65 + 8xy - x^2y^2$.
52. $110 - x - x^2$.
53. $98 - 7x - x^2$.
54. $380 - x - x^2$.
55. $120 - 7ax - a^2x^2$.
56. $105 + 16cy - c^2y^2$.
57. $(x+y)^2 + 6(x+y) + 8$.
58. $(a-b)^2 + 8(a-b) + 15$.

140. Trinomials of the type-form $ax^2 + bx + c$.

Multiplying and dividing $ax^2 + bx + c$ by a, we obtain

$$ax^2 + bx + c \equiv [(ax)^2 + b(ax) + ac] \div a. \qquad (1)$$

By § 139, the trinomial in brackets can be factored by finding the two factors of ac whose sum is b.

Ex. 1. $3x^2 - 16x + 5 \equiv [(3x)^2 - 16(3x) + 15] \div 3$
$\equiv (3x - 15)(3x - 1) \div 3$ § 139
$\equiv (x - 5)(3x - 1).$

Ex. 2. $5x^2 + 32x - 21 \equiv [(5x)^2 + 32(5x) - 105] \div 5$
$\equiv (5x + 35)(5x - 3) \div 5$
$\equiv (x + 7)(5x - 3).$

Ex. 3. $3x^2 - 17xy + 10y^2 \equiv [(3x)^2 - 17y(3x) + 30y^2] \div 3$
$\equiv (x - 5y)(3x - 2y).$

Exercise 55.

Factor each of the following expressions:

1. $2x^2 + 3x + 1.$
2. $3x^2 + 5x + 2.$
3. $3x^2 + 10x + 3.$
4. $3x^2 + 8x + 4.$
5. $2x^2 + 7x + 6.$
6. $2x^2 + 11x + 5.$
7. $5x^2 + 11x + 2.$
8. $2x^2 + 3x - 2.$
9. $4x^2 + 11x - 3.$
10. $2x^2 + 15x - 8.$
11. $3x^2 + 7x - 6.$
12. $2x^2 + x - 28.$
13. $3x^2 + 13x - 30.$
14. $6x^2 + 7x - 3.$
15. $3a^2x^2 + 23ax + 14.$
16. $3a^2x^2 + 19ax - 14.$
17. $6a^2x^2 - 31ax + 35.$
18. $3x^2 + 41x + 26.$
19. $4x^2 + 23x + 15.$
20. $3x^2 - 13x + 14.$
21. $2x^2 - 5xy - 3y^2.$
22. $3x^2 - 17xy + 10y^2.$
23. $12x^2 - 23xy + 10y^2.$
24. $24x^2 - 29xy - 4y^2.$

Factor each of the following *miscellaneous* expressions:

25. $2x(n - 1) - 2(1 - n).$
26. $cy^m - ay^{m+2} + ny^{m+1}.$
27. $9a^4 + 16b^4 - 24a^2b^2.$
28. $7x^2 - 15xy - 18y^2.$
29. $5x(a-2y) - 2(2y-a).$
30. $x^2/9 + y^2/4 + xy/3.$

31. $(x-3)^2+4(3-x)+4.$
32. $132x^2+x-1.$
33. $ax^2+(a+b)x+b.$
34. $x(x-a)^2-y(x-a).$
35. $x^{m-1}-3x^{m+1}-5x^{m+2}.$
36. $121x^2+81y^2+198xy.$
37. $12x^2+50x-50.$
38. $ax^2+(a-b)x-b.$
39. $x^2+2xy-4xz-4yz+4z^2.$
40. $5y^{m-1}-3y^{m+1}+4y^{m+2}.$
41. $(a+b)^2+5(a+b)-24.$
42. $(x-y)^2-4(x-y)-21.$

141. Binomials of the type-form $a^n - b^n$, where n is even.

The converse of the identity in § 122 is

$$a^2-b^2 \equiv (a+b)(a-b). \tag{1}$$

That is, *the difference of the squares of any two numbers is equal to the product of the sum and the difference of the numbers.*

Ex. 1. $9a^6b^6-4c^2 \equiv (3a^3b^3)^2-(2c)^2$
$\equiv (3a^3b^3+2c)(3a^3b^3-2c)$ by (1)

The letters a and b in (1) stand for any expressions.

Ex. 2. $a^2-4ay+4y^2-9c^2 \equiv (a-2y)^2-(3c)^2$
$\equiv (a-2y+3c)(a-2y-3c)$ by (1)

Ex. 3. $9x^2+12ab-9a^2-4b^2 \equiv (3x)^2-(3a-2b)^2$
$\equiv (3x+3a-2b)(3x-3a+2b).$

In factoring a given expression, it may be necessary to use the same principle two or more times in succession as below:

Ex. 4. $(x^2-y^2+z^2)^2-4x^2z^2$
$\equiv (x^2-y^2+z^2+2xz)(x^2-y^2+z^2-2xz)$
$\equiv [(x+z)^2-y^2][(x-z)^2-y^2]$
$\equiv (x+z+y)(x+z-y)(x-z+y)(x-z-y).$

Whenever n is even, a^n-b^n should be factored as the difference of two squares.

Ex. 5. $x^4-a^4 \equiv (x^2)^2-(a^2)^2$
$\equiv (x^2+a^2)(x^2-a^2)$
$\equiv (x^2+a^2)(x+a)(x-a).$

Exercise 56.

Factor each of the following expressions:

1. $a^2 - 9$.
2. $25a^2 - b^2$.
3. $16 - b^2$.
4. $x^2 - 9y^2$.
5. $64x^2 - 49b^2$.
6. $9a^2 - 16b^2$.
7. $81y^2 - 9x^2$.
8. $36x^2 - 49y^2$.
9. $4a^2b^2 - 9c^2$.
10. $4xy^2 - 9x^3$.
11. $8ab^2 - 18a^3$.
12. $108x^3 - 3x^5$.
13. $7x^5 - 28x^9$.
14. $32xy^3 - 8x^3y$.
15. $7xyz^2 - 7x^3y^3$.
16. $a^2 + 2ab + b^2 - c^2$.
17. $a^2 - 2ab + b^2 - c^2$.
18. $a^2 - b^2 - 2bc - c^2$.
19. $a^2 - b^2 + 2bc - c^2$.
20. $x^2 + 4xy - a^2 + 4y^2$.
21. $x^2 - 1 + 10cx + 25c^2$.
22. $1 + 2ab - a^2 - b^2$.
23. $49x^2 - 1 + 14xy + y^2$.
24. $a^2 - 16x^2 + 6ab + 9b^2$.
25. $x^2 - 9y^2 + 10ax + 25a^2$.
26. $b^2 - a^2 - 4x^2 + 4ax$.
27. $9c^2 - 4x^2 - 9a^2 + 12ax$.
28. $4a^2 - y^2 - 9z^2 + 6yz$.
29. $c^2 - 25a^2 - 9b^2 + 30ab$.
30. $a^2 + b^2 + 2ab - c^2 - d^2 - 2cd$.
31. $a^2 + b^2 - 2ab - x^2 - y^2 - 2xy$.
32. $m^2 + n^2 - 2mn - a^2 - b^2 + 2ab$.
33. $a^2 + n^2 - 2an - b^2 - m^2 - 2bm$.
34. $16a^2 + 8ax + x^2 - 2by - b^2 - y^2$.
35. $9a^2 + 12ab + 4b^2 - (c + x - 2y)^2$.
36. $(a + b + c)^2 - x^2 - y^2 + 2xy$.
37. $(x + 3y)^2 - 4y^2$.
38. $9a^2 - (3a - 5b)^2$.
39. $(5x + 2y)^2 - (3x - y)^2$.
40. $(2x + a - 3)^2 - (3 - 2x)^2$.
41. $\frac{5}{4}xa^4 - \frac{5}{25}xb^6$.
42. $\frac{2}{4}a^3x^2 - \frac{2}{9}ay^4$.
43. $16x^4 - y^4$.
44. $a^4 - 81$.
45. $5 - 80x^4$.
46. $a^4x^4 - 16b^4y^4$.
47. $x^{2n} - y^{2n}$.
48. $x^{2n+2} - y^{2n-2}$.
49. $x^{4n} - y^{4n}$.
50. $9x^n - x^{n+2}$.
51. $4x^{3n+3} - x^{n+1}$.
52. $x^{2n+3}y^{2n} - x^5y^2$.
53. $x^2 + y^2 + z^2 + 2xy - 2xz + 2yz - 16$.
54. $a^4 + 4b^2 + 9c^2 - 4ab + 6ac - 12bc - x^2 - 2xy - y^2$.

142. Binomials of the type-form $a^n - b^n$, where n is odd.

When $n = 3$, by § 129 we have

$$a^3 - b^3 \equiv (a - b)(a^2 + ab + b^2).$$

Ex. 1. $243 - 8a^3 \equiv (7)^3 - (2a)^3$
$$\equiv (7 - 2a)[7^2 + 7(2a) + (2a)^2]$$
$$\equiv (7 - 2a)(49 + 14a + 4a^2).$$

Ex. 2. $125 - 8a^6b^6 \equiv (5)^3 - (2a^2b^2)^3$
$$\equiv (5 - 2a^2b^2)[5^2 + 5(2a^2b^2) + (2a^2b^2)^2]$$
$$\equiv (5 - 2a^2b^2)(25 + 10a^2b^2 + 4a^4b^4).$$

Ex. 3. $(1 - 2x)^3 - 64x^3 \equiv (1 - 2x)^3 - (4x)^3$
$$\equiv (1 - 2x - 4x)[(1 - 2x)^2 + (1 - 2x)(4x) + (4x)^2]$$
$$\equiv (1 - 6x)(1 + 12x^2).$$

When $n = 5$, by § 129 we have

$$a^5 - b^5 \equiv (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4).$$

Ex. 4. $2a^5 - 64b^5 \equiv 2[a^5 - (2b)^5]$
$$\equiv 2(a - 2b)[a^4 + a^3(2b) + a^2(2b)^2 + a(2b)^3 + (2b)^4]$$
$$\equiv 2(a - 2b)(a^4 + 2a^3b + 4a^2b^2 + 8ab^3 + 16b^4).$$

From identity (1) in § 129, we have

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1}),$$

when n is any positive integer.

143. Binomials of the type-form $a^n + b^n$, where n is odd.

When $n = 3$, by § 130 we have

$$a^3 + b^3 \equiv (a + b)(a^2 - ab + b^2).$$

Ex. 1. $8x^3 + 27y^3 \equiv (2x)^3 + (3y)^3.$
$$\equiv (2x + 3y)[(2x)^2 - (2x)(3y) + (3y)^2]$$
$$\equiv (2x + 3y)(4x^2 - 6xy + 9y^2).$$

When $n = 5$, by § 130 we have

$$a^5 + b^5 \equiv (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4).$$

From identity (2) in § 130 we have, when n is odd,

$$a^n + b^n \equiv (a + b)(a^{n-1} - a^{n-2}b + \cdots - ab^{n-2} + b^{n-1}).$$

Exercise 57.

Factor each of the following expressions:

1. $x^3 - 1$.
2. $27 - x^3$.
3. $a^3 - 8b^3$.
4. $125 - a^3b^3$.
5. $x^3 + 1$.
6. $y^3 + 27$.
7. $8x^3 + 64$.
8. $343 - 8a^3$.
9. $216 - x^3$.
10. $27n^3 + 1$.
11. $8x^3 - 27a^6$.
12. $x^3b^3 - a^3/8$.
13. $40a^3 - 135b^3$.
14. $27n^3 + 64c^3$.
15. $y^5 - 1$.
16. $x^5 - 32$.
17. $32a^5 + 1$.
18. $a^5b^5 + 243$.
19. $1024x^5 - 32y^5$.
20. $x^7 - y^7$.
21. $x^7 - 1$.
22. $x^7 + 128$.
23. $1 - (x + y)^3$.
24. $x^6 - y^6$.

When n is even, $x^n - y^n$ should *first* be factored as the difference of two squares (§ 141).

$$x^6 - y^6 \equiv (x^3 - y^3)(x^3 + y^3) \qquad \text{§ 141}$$
$$\equiv (x - y)(x^2 + xy + y^2)(x + y)(x^2 - xy + y^2).$$

25. $x^6 - 1$.
26. $a^6 - 64$.
27. $x^6 - 64y^6$.
28. $x^6y^6 - a^6b^6$.
29. $x^4 - 16b^4$.
30. $16x^4 - 81a^4$.
31. $81a^4x^4 - 1$.
32. $a^6 - 729b^6$.
33. $64x^6 - 729y^6$.
34. $(3 + 2a)^3 - 64$.
35. $a^3 - (x + y)^3$.
36. $x^5 - (a - b)^5$.
37. $a^{5n} - (x - 2b)^5$.
38. $a^4x^4 - (b - c)^4$.
39. $x^6y^6 - (xy + 1)^6$.
40. $16x^8 - (y + 2z)^4$.
41. $27x^{3n} - (a + b)^6$.

144. A trinomial of the type-form $\boldsymbol{a^4 + ha^2b^2 + b^4}$ can be factored by writing it as *the difference of two squares.*

Note. The two factors of $a^4 + ha^2b^2 + b^4$ are real and equal when $h = 2$; real and unequal when $h < 2$, and complex when $h > 2$.

In all the examples given the factors are real and unequal, but as some of them involve surds this article and the next should be omitted until Chapter XVII. has been studied.

Ex. 1. Factor $m^4 + m^2n^2 + n^4$.

Adding $m^2n^2 - m^2n^2$, we obtain

$$\begin{aligned} m^4 + n^4 + m^2n^2 &\equiv m^4 + n^4 + 2\,m^2n^2 - m^2n^2 \\ &\equiv (m^2 + n^2)^2 - (mn)^2 \\ &\equiv (m^2 + n^2 + mn)(m^2 + n^2 - mn). \end{aligned}$$

Ex. 2.
$$\begin{aligned} m^4 - 5\,m^2n^2 + n^4 &\equiv m^4 + n^4 - 2\,m^2n^2 - 3\,m^2n^2 \\ &\equiv (m^2 - n^2)^2 - (mn\sqrt{3})^2 \\ &\equiv (m^2 - n^2 + mn\sqrt{3})(m^2 - n^2 - mn\sqrt{3}). \end{aligned}$$

Or,
$$\begin{aligned} m^4 - 5\,m^2n^2 + n^4 &\equiv m^4 + n^4 + 2\,m^2n^2 - 7\,m^2n^2 \\ &\equiv (m^2 + n^2 + mn\sqrt{7})(m^2 + n^2 - mn\sqrt{7}). \end{aligned}$$

Ex. 3.
$$\begin{aligned} 4\,x^4 + 9\,a^4 - 21\,a^2x^2 &\equiv 4\,x^4 + 9\,a^4 - 12\,a^2x^2 - 9\,a^2x^2 \\ &\equiv (2\,x^2 - 3\,a^2)^2 - (3\,ax)^2 \\ &\equiv (2\,x^2 - 3\,a^2 + 3\,ax)(2\,x^2 - 3\,a^2 - 3\,ax). \end{aligned}$$

Exercise 58.

Factor each of the following expressions:

1. $x^4 + x^2 + 1.$
2. $x^4 - 3\,x^2 + 9.$
3. $x^4 + 9\,x^2 + 25.$
4. $x^4 + 9\,x^2 + 25.$
5. $x^4 - 11\,a^2x^2 + a^4.$
6. $x^4 + (4 - c^2)\,x^2y^2 + 4\,y^4.$
7. $(x + y)^4 + (x + y)^2 + 1.$
8. $9\,x^4 + 3\,x^2y^2 + 4\,y^4.$
9. $25\,x^4 - 44\,x^2y^2 + 16\,y^4.$
10. $4\,x^4 - 4\,x^2y^2 + 9\,y^4.$
11. $9\,x^4 - 12\,x^2y^2 + 16\,y^4.$
12. $16\,x^4 - x^2y^2 + y^4.$
13. $25\,x^4 - 29\,x^2y^2 + 4\,y^4.$
14. $x^4 - x^2y^2 + y^4.$
15. $x^4 + x^2y^2 + y^4.$
16. $x^8 + x^4y^4 + y^8.$

145. Binomials of the type-form $a^n + b^n$, where n is even.

(i) $a^4 + b^4 \equiv a^4 + 2a^2b^2 + b^4 - 2a^2b^2$
$$\equiv (a^2+b^2)^2 - (ab\sqrt{2})^2$$
$$\equiv (a^2+b^2+ab\sqrt{2})(a^2+b^2-ab\sqrt{2}).$$

This method can be employed whenever n is a multiple of 4, as when n is 8, 12, 16, etc.

(ii) $a^6 + b^6 \equiv (a^2)^3 + (b^2)^3$
$$\equiv (a^2+b^2)[a^4 - a^2b^2 + b^4]$$
$$\equiv (a^2+b^2)[(a^2+b^2)^2 - (ab\sqrt{3})^2]$$
$$\equiv (a^2+b^2)(a^2+b^2+ab\sqrt{3})(a^2+b^2-ab\sqrt{3}).$$

This method can be employed when n is even and one of its two factors is *odd*, as when n is 10, 12, 14, etc.

Exercise 59.

Factor each of the following expressions:

1. x^4+1.
2. $x^4+c^4y^4$.
3. $16x^4+a^4$.
4. x^8-1.
5. x^8-a^8.
6. x^6+1.
7. x^6+64.
8. $x^6+c^6y^6$.
9. x^8+a^8.
10. x^8+1.
11. $x^{10}+a^{10}$.
12. $x^{12}+a^{12}$.

146. Perfect cubes. The converse of identity (1) in § 124 is
$$a^3 + 3a^2b + 3ab^2 + b^3 \equiv (a+b)^3.$$

Hence, if the four terms of the cube of a binomial are arranged according to the powers of some letter, their extreme terms are the cubes of the terms of the binomial.

E.g., if $64a^3 - 144a^2b + 108ab^2 - 27b^3$ is a perfect cube, it is the cube of $4a - 3b$; for when its four terms are arranged in descending powers of a, the extreme terms are the cubes of $4a$ and $-3b$ respectively.

The expression is a perfect cube; for
$$(4a-3b)^3 \equiv 64a^3 - 144a^2b + 108ab^2 - 27b^3.$$

If a perfect cube which contains only *three* different powers of some letter is arranged according to the powers of that letter, its factors will often become obvious.

E.g., if we arrange the expression,

$$a^3 + b^3 + c^3 + 3a^2b + 3a^2c + 3ab^2 + 3ac^2 + 6abc + 3b^2c + 3bc^2,$$

according to the three different powers of a, we have

$$a^3 + 3a^2(b + c) + 3a(b^2 + c^2 + 2ab) + (b^3 + 3b^2c + 3bc^2 + c^3),$$

or
$$a^3 + 3a^2(b + c) + 3a(b + c)^2 + (b + c)^3,$$

which is seen to be $(a + b + c)^3$.

Exercise 60.

Factor each of the following expressions:

1. $a^3 + 3a^2 + 3a + 1$.
2. $x^3 + 6x^2 + 12x + 8$.
3. $8m^3 - 12m^2 + 6m - 1$.
4. $a^3x^3 - 3a^2x^2y^2 + 3axy^4 - y^6$.
5. $64a^3 + 108ab^2 - 144a^2b - 27b^3$.
6. $x^3 - 24x^2y + 192xy^2 - 512y^3$.
7. $a^3 + 6a^2b - 3a^2c + 12ab^2 - 12abc + 3ac^2 + 8b^3 - 12b^2c + 6bc^2 - c^3$.
8. $\dfrac{x^3}{8} - \dfrac{3x^2}{4} + \dfrac{3x}{2} - 1$.
9. $8x^3 - 4x^2y^2 + \frac{2}{3}xy^4 - \dfrac{y^6}{27}$.
10. $24b^2x^3 - 36b^2x^2 + 18b^2x - 3b^2$.
11. $a^2 + 2ab + 4c^2 + 4ac + 4bc + b^2$.
12. $2ax^4 + 4ax^2y^2 - 4ay^2z^2 + 2az^4 + 2ay^4 - 4ax^2z^2$.
13. $3bx^4 - 6bx^2y + 12a^2b - 12aby + 3by^2 + 12abx^2$.
14. $a^2x^3 - 9a^2x^2y + 27a^2xy^2 - 27a^2y^3$.
15. $x^3 + 3x^2y + 3xy^2 - 3ax^2 + y^3 - 3ay^2 - 6axy + 3a^2y + 3a^2x - a^3$.

147. Summary. To factor any given expression by the foregoing methods, the pupil should first note whether the expression is in any one of the following forms:

(i) A sum of terms having a common factor. § 136

(ii) A perfect power. §§ 137, 138, 146

(iii) A difference of squares. § 141

(iv) The type-form

$$ax^2 + bx + c \text{ or } x^2 + px + q. \qquad \S\S\ 139, 140$$

(v) The type-form $a^n - b^n$ or $a^n + b^n$, n odd. §§ 142, 143

(vi) The type-form

$$a^4 + ha^2b^2 + b^4 \text{ or } a^n + b^n, n \text{ even}. \qquad \S\S\ 144, 145$$

When a factorable expression has no one of these forms, our first aim is to reduce it to one of them. In this reduction much will in the end depend upon the ingenuity of the student. No definite directions which are applicable to all cases can be given. The two following devices will in many cases prove useful:

(i) The factors of an expression will frequently become obvious when the *expression is arranged in ascending or descending powers of one of its letters*, particularly when the expression contains only one power of that letter.

Ex. 1. Factor $ax + by + bx + ay$.

Arranging in powers of x, we have

$$\begin{aligned} ax + by + bx + ay &\equiv (a + b)x + (a + b)y \\ &\equiv (a + b)(x + y). \end{aligned}$$

Ex. 2. Factor $ax^3 - x - a + 1$.

Arranging in powers of a, we have

$$\begin{aligned} ax^3 - x - a + 1 &\equiv (x^3 - 1)a - (x - 1) \\ &\equiv (x - 1)[a(x^2 + x + 1) - 1] \\ &\equiv (x - 1)(ax^2 + ax + a - 1). \end{aligned}$$

Ex. 3. Factor $a^2(x-y)+x^2(y-a)+y^2(a-x)$.

Arranging in powers of a, we have

$$\begin{aligned}\text{the given expression } &\equiv a^2(x-y)-a(x^2-y^2)+xy(x-y)\\ &\equiv (x-y)[a^2-(x+y)a+xy]\\ &\equiv (x-y)(a-x)(a-y).\end{aligned}$$

(ii) Another device consists in adding to the given expression some form of zero; as, y^2-y^2, or $-1+1$.

Ex. 1. Factor $x^2-3y^2-z^2-2xy+4yz$.

Arranging in descending powers of x and adding y^2-y^2, we obtain

$$\begin{aligned}\text{the given expression } &\equiv x^2-2xy+y^2-(4y^2+z^2+4yz)\\ &\equiv (x-y)^2-(2y-z)^2\\ &\equiv (x-y+2y-z)(x-y-2y+z)\\ &\equiv (x+y-z)(x-3y+z).\end{aligned}$$

Ex. 2. Factor x^3-3x+2.

Adding $-1+1$, we obtain

$$\begin{aligned}x^3-3x+2 &\equiv (x^3-1)-3(x-1)\\ &\equiv (x-1)(x^2+x+1-3)\\ &\equiv (x-1)(x-1)(x+2).\end{aligned}$$

Ex. 3. Factor x^3-3x^2+4.

Adding x^2-x^2, or putting $-2x^2-x^2$ for $-3x^2$, we obtain

$$\begin{aligned}x^3-3x^2+4 &\equiv x^3-2x^2-x^2+4\\ &\equiv (x-2)x^2-(x^2-4)\\ &\equiv (x-2)(x^2-x-2)\\ &\equiv (x-2)(x-2)(x+1).\end{aligned}$$

Exercise 61.

Factor each of the following expressions:

1. $a^2+ab+ac+bc$.
2. $a^2c^2+acd+abc+bd$.
3. $a^2+3a+ac+3c$.
4. $mx-my-nx+ny$.
5. $3ax-bx-3ay+by$.
6. $6x^2+3xy-2ax-ay$.

7. $ax^2 - 3bxy - axy + 3by^2$.

8. $2ax^2 + 3axy - 2bxy - 3by^2$.

9. $amx^2 + bmxy - anxy - bny^2$.

10. $ax - bx + by + cy - cx - ay$.

11. $a^2x + abx + ac + aby + b^2y + bc$.

12. $x^3 + x^2 - 4x - 4$.

13. $5x^3 - x^2 - 5x + 1$.

14. $ax^3 + bx^3 + a + b$.

15. $ax^2 + by^2 + (a+b)xy$.

16. $a^2b^2 + a^2 + b^2 + 1$.

17. $a^4 + a^2b^2 - b^2c^2 - c^4$.

18. $a^2 - a - c^2 + c$.

19. $a^2 - b^2 - (a-b)^2$.

20. $a^2 - b^2 + bc - ca$.

21. $ax^3 + x^3 + a + 1$.

22. $x^3 - 5x^2 + x - 5$.

23. $2x^3 - 3x^2 - 2x + 3$.

24. $x^3 + bx^2 - a^2x - a^2b$.

25. $a^2b^2 - a^2 - b^2 + 1$.

26. $bx^3 + ax^2 + bx + a$.

27. $x^2 - y^2 + xz - yz$.

28. $1 + bx - (a^2 + ab)x^2$.

29. $a^2c^2 + acd + abc + bd$.

30. $ac + bd - ad - bc$.

31. $ac^2 + bd^2 - ad^2 - bc^2$.

32. $a^2x - b^2x + a^2y - b^2y$.

33. $a^3x^2 - c^3x^2 - a^3y^2 + c^3y^2$.

34. $a^2x^6 - a^2y^6 - b^2x^6 + b^2y^6$.

35. $acx^2 - bcx + adx - bd$.

36. $c^5d^3 - c^2 - a^2c^3d^3 + a^2$.

37. $1 - abx^3 + (b - a^2)x^2$.

38. $a^2 - b^2 + c^2 - d^2 - 2(ac - bd)$.

39. $4a^2b^2 - (a^2 + b^2 - c^2)^2$.

40. $(a^2 - b^2 + c^2 - d^2)^2 - (2ac - 2bd)^2$.

41. $x^4 + x^3y + xz^3 + yz^3$.

42. $x(x+z) - y(y+z)$.

43. $x^4 - 14x^2y^2 + y^4$.

44. $x^2y^2 - x^2z^2 - y^2z^2 + z^4$.

45. $1 - 2ax - (c - a^2)x^2 + acx^3$.

46. $ax(y^3 + b^3) + by(bx^2 + a^2y)$.

47. $2x^3 - 4x^2y - x^2z + 2xy^2 + 2xyz - y^2z$.

48. $(x^2 + 4x)^2 - 2(x^2 + 4x) - 15$.

49. $(a^2 - 2a)^2 - 2(a^2 - 2a) - 3$.

50. $(x^2 + 4x + 8)^2 + 3x(x^2 + 4x + 8) + 2x^2$.

51. $x^3 - 6x^2 + 16$.

52. $x^3 - 15x^2 + 250$.

53. $x^4y^4 + 13x^2y^4 + 49y^4$.

54. $36x^4y^4 + 3xy^4 + y^4$.

55. Resolve $a^9 - 64a^3 - a^6 + 64$ into six factors.

56. Resolve $x^7 + x^4 - 16x^3 - 16$ into five factors.

57. Resolve $16x^7 - 81x^3 - 16x^4 + 81$ into five factors.

58. Resolve $x^9 + x^6 + 64x^3 + 64$ into four factors.

59. Resolve $x^9 + x^3y^3 - 8x^6y^3 - 8y^9$ into four factors.

60. Factor $a^2(b - c) + b^2(c - a) + c^2(a - b)$.

148. Formation of equations with given roots.

The linear equation whose root is 4 is evidently $x - 4 = 0$. (1)

The linear equation whose root is -2 is evidently $x + 2 = 0$. (2)

Multiplying together the corresponding members of (1) and (2), we obtain the quadratic equation $(x - 4)(x + 2) = 0$. (3)

When $x = 4$, (3) becomes the identity $(4 - 4)(4 + 2) = 0$.

When $x = -2$, (3) becomes the identity $(2 - 4)(-2 + 2) = 0$.

No other value of x will render either factor in (3) equal to 0.

Hence 4 and -2 are the two and only roots of (3).

The quadratic equation (3) therefore is equivalent to, *i.e.* has the same roots as, the two linear equations (1) and (2) together.

This example illustrates the following principle:

The linear equations

$$x - a = 0, \quad x - b = 0, \quad x - c = 0, \quad \cdots \qquad (1)$$

are jointly equivalent to the equation

$$(x - a)(x - b)(x - c) \cdots = 0. \qquad (2)$$

Proof. The root of any one of the equations in (1) renders one of the factors in (2) zero; hence by § 74 it satisfies (2).

Conversely each root of (2) must render one factor of its first member zero, and hence be a root of one of the equations (1).

Moreover, equations (1) have the same number of roots as equation (2).

Hence the linear equations (1) are jointly equivalent to equation (2).

Ex. Form an equation whose roots are 1, -3, and 4.

The linear equations whose roots are 1, -3, and 4, respectively are

$$x - 1 = 0, \quad x + 3 = 0, \quad x - 4 = 0. \qquad (1)$$

By § 148 the equation which is equivalent to equations (1) is

$$(x - 1)(x + 3)(x - 4) = 0,$$

or

$$x^3 - 2x^2 - 11x + 12 = 0.$$

Observe, (i) that the second member of each of the equations (1) and (2) is 0, (ii) that equation (2) is formed from equations (1) by multiplying together their corresponding members, and (iii) that equations (1) are formed from (2) by putting each factor of its first member equal to 0.

Exercise 62.

Form the equation whose roots are

1. $+4, +3$.
2. $-4, +3$.
3. $2, 3$.
4. $2, -3$.
5. $-2, 3$.
6. $-2, -3$.
7. $-7, 4$.
8. $1, 2, 3$.
9. $1, -2, -3$.
10. $-1, -2, -3$.
11. $3, -4, 5$.
12. $1, -2, 3, -4$.

149. To solve a quadratic or higher equation we must find its *equivalent linear equations.*

For use in solving equations the principle proved in § 148 can be stated as follows:

If one member of an equation is zero and the other member is the product of two or more integral factors, the equations formed by putting each of these factors equal to zero are together equivalent to the given equation.

E.g., the equation $(x-2)(x+3)(x-4)=0$
is equivalent to the three linear equations,

$$x-2=0,\ x+3=0,\ x-4=0.$$

Ex. 1. Solve the equation $x^2=4x+12$. (1)

Transpose, $x^2-4x-12=0.$

Factor the first member, $(x+2)(x-6)=0.$ (2)

Equation (2) is equivalent to the two linear equations,

$$x+2=0,\ x-6=0.$$

Hence, the roots of (2), or (1), are -2 and 6.

Ex. 2. Solve the higher equation $x^3+x^2=6x$. (1)

Transpose, $x^3+x^2-6x=0.$

Factor, $x(x-2)(x+3)=0.$ (2)

Equation (2) is equivalent to the three linear equations,

$$x=0,\ x-2=0,\ x+3=0.$$

Hence, the roots of (2), or (1), are 0, 2, and -3.

Ex. 3. Solve the equation $9x^3=4x$. (1)

Transpose, $9x^3-4x=0.$

Factor, $x(3x+2)(3x-2)=0.$ (2)

Equation (2) is equivalent to the three linear equations,

$$x=0,\ 3x+2=0,\ 3x-2=0.$$

Hence, the roots of (2), or (1), are 0, $-\frac{2}{3}$, and $\frac{2}{3}$.

Ex. 4. Solve the equation $4x^4 + 9 = 37x^2$. (1)

Transpose, $4x^4 - 37x^2 + 9 = 0$.

Factor, $(2x-6)(2x+6)(2x-1)(2x+1) = 0$. (2)

Equation (2) is equivalent to the four linear equations,

$$2x - 6 = 0,\ 2x + 6 = 0,\ 2x - 1 = 0,\ 2x + 1 = 0.$$

Hence the roots of (2) or (1) are 3, -3, $\frac{1}{2}$, and $-\frac{1}{2}$.

These examples illustrate the following rule for solving a quadratic or higher equation in one unknown:

Transpose all the terms to one member.

Resolve this member into its linear factors in the unknown.

Solve the equations formed by equating to zero each of these linear factors.

The problem of *solving* an equation is the *converse* to that of *forming* an equation with given roots.

If we multiply together the corresponding members of equations

$$x - 3 = 1 \text{ and } x + 3 = 16, \quad (1)$$

we obtain $x^2 - 9 = 16$, or $x^2 - 25 = 0$. (2)

The roots of equations (1) are 4 and 13, and the roots of (2) are 5 and -5.

Hence both roots of equations (1) are lost by multiplying together their corresponding members.

Putting equations (1) in the form

$$x - 4 = 0 \text{ and } x - 13 = 0,$$

and then multiplying them together, we obtain an equation equivalent to equations (1).

This illustrates the importance of the *form* of the equations in § 148

Exercise 63.

Solve each of the following equations:

1. $x^2 - 7x = 0$.
2. $x^2 + 9x = 0$.
3. $x^2 = 4x + 12$.
4. $x^2 + 12x = -35$.
5. $x^2 = 6x + 91$.
6. $x^2 + 12 = 7x$.

7. $x^2 + 20 = 12x$.
8. $x^2 + 20 = 9x$.
9. $x^2 + 28 = 11x$.
10. $x^2 + 150 = 25x$.
11. $3x^2 = 10x - 3$.
12. $3x^2 + 11x = 20$.
13. $4x^2 + 21x = 18$.
14. $3x^2 - 2x = 96$.
15. $15x^2 + 4x = 3$.
16. $6x^2 - 7x = 3$.
17. $19x = 4 - 5x^2$.
18. $5x^2 - 4x = 33$.
19. $x^2 + ax = 42a^2$.
20. $x^2 - 20ax = 96a^2$.
21. $8x^2 + x = 30$.
22. $x + 22 = 6x^2$.
23. $21 + x = 2x^2$.
24. $3x^2 + 35 = 22x$.
25. $6x^2 + 55x = 50$.
26. $6x^2 + 6 = 13x$.
27. $19x^2 - 39x = -2$.
28. $15x^2 - 2ax = a^2$.
29. $17x^2 + 8 = 70x$.
30. $21x^2 + 10x = -1$.
31. $6x^2 = 11kx + 7k^2$.
32. $x^2 - 23ax = -132a^2$.
33. $2x^2 - 3a^2 = 5ax$.
34. $12x^2 + 3a^2 = 13ax$.
35. $132x^2 + x = 1$.
36. $x^2 + 600a^2 = -49ax$.
37. $x^3 - 3x^2 = 10x$.
38. $16x^3 + 3x = 16x^2$.
39. $110x^3 + x = 21x^2$.
40. $5x^3 = 8x^2 + 21x$.
41. $32x - 3x^3 = 10x^2$.
42. $x^3 + 2a^2x = 3ax^2$.
43. $x^3 - x^2 + 9 = 9x$.
44. $x^3 + 2x^2 - 16x = 32$.
45. $x^4 - 26x^2 + 25 = 0$.
46. $x^4 + 36 = 13x^2$.
47. $36x^4 + 1 = 13x^2$.
48. $x^4 - a^2x^2 + 4a^2c^2 = 4c^2x^2$.
49. $x^2 + 2a^2 = 3ax$.
50. $35b^2 = 9x^2 + 6bx$.
51. $x^2 - 2ax + 4ab = 2bx$.
52. $3x^2 - 2ax - bx = 0$.
53. $x^2 - 2ax + 8x = 16a$.
54. $36x^2 - 35b^2 = 12bx$.
55. $x^2 + 2(b - c)x + c^2 = 2bc$.
56. $x^2 - 2(a - b)x + b^2 = 2ab$.
57. $(a - x)^3 + (x - b)^3 = (a - b)^3$.
58. $x^3 + x^2 = 4x + 4$.

59. $5x^3 - x^2 = 5x - 1.$

60. $x^2 - x = c^2 - c.$

61. $x^2 - b^2 = cx - bc.$

62. $2x^3 - 3x^2 = 2x - 3.$

63. $x^3 + bx^2 = a^2x + a^2b.$

64. $x^3 + 5 = 5x^2 + x.$

65. $bx^3 + ax^2 = bx + a.$

66. $x^3 - 3x^2 = 4x - 12.$

67. $x^4 + 36 = 13x^2.$

68. $4x^4 + 9 = 13x^2.$

69. $x^3 + 2x^2 = 16x + 32.$

70. $9x^3 + 27x^2 = x + 3.$

71. Find two numbers one of which is three times the other and whose product is 243.

72. Find two numbers whose sum is 18 and whose product is 77.

73. A certain number is subtracted from 36, and the same number is also subtracted from 30; and the product of the remainders is 891. Find the number.

74. A rectangular court is 10 rods longer than it is broad; its area is 375 square rods. Find its length and breadth.

75. How many children are there in a family, when eleven times the number is greater by five than twice the square of the number?

76. Eleven times the number of yards in the length of a rod is greater by five than twice the square of the number of yards. How long is the rod?

77. The square of the number of dollars a man possesses is greater by 1000 than thirty times the number. How much is the man worth?

Ans. The man may have $ 50 or he may owe $ 20.

78. Find two numbers the sum of whose squares is 74, and whose sum is 12.

CHAPTER XI

HIGHEST COMMON FACTORS AND LOWEST COMMON MULTIPLES

150. A **common factor** of two or more expressions is an expression which will exactly divide each of them.

E.g., $a - x$ is a common factor of $b(a - x)$ and $a^2 - x^2$.

151. Two or more expressions are said to be **prime** to one another, when they have no common integral factor except 1.

E.g., xy and vz, $3a^2b$ and $7c^3$, or $x^2 + y^2$ and $x^2 - y^2$, are prime to each other.

152. The **highest common factor** (H. C. F.) of two or more integral literal expressions is the expression of highest degree which will exactly divide each of them.

The *numeral* factor of the H. C. F. is the greatest common measure (G. C. M.) of the numeral factors of the given expressions.

E.g., $x^2y^2z^3$ is the H. C. F. of $x^3y^2z^4$ and $x^2y^4z^3$.

Again, $10x^3yz$ is the H. C. F. of $20x^4yz$ and $30x^3y^2z^3$.

153. H. C. F. by factoring.

Ex. 1. Find the H. C. F. of $6a^2b^3c^3d^2$, $4a^3c^5d$, and $8a^4bc^6d^3$.

The H. C. F. of these expressions cannot contain a higher power of a than a^2, a higher power of c than c^3, and a higher power of d than d; and the G. C. M. of the numeral factors is 2.

Hence the H. C. F. of these expressions is $2a^2c^3d$.

Observe that the power of each base in the H. C. F. is the *lowest* power to which it occurs in any of the given expressions.

Ex. 2. Find the H. C. F. of $a^4b^2 - a^2b^4$ and $a^4b^3 + a^3b^4$.

$$a^4b^2 - a^2b^4 \equiv a^2b^2(a+b)(a-b);$$

$$a^4b^3 + a^3b^4 \equiv a^3b^3(a+b).$$

$$\therefore \text{ H. C. F.} \equiv a^2b^2(a+b).$$

Ex. 3. Find the H. C. F. of $3\,a^4 + 15\,a^3b - 72\,a^2b^2$, $6\,a^3 - 30\,a^2b + 36\,ab^2$, and $8\,b^5 - 16\,a^4b - 24\,a^3b^2$.

$$3\,a^4 + 15\,a^3b - 72\,a^2b^2 \equiv 3 \cdot a^2(a+8\,b)(a-3\,b);$$

$$6\,a^3 - 3\,a^2b + 36\,ab^2 \equiv 6 \cdot a(a-2\,b)(a-3\,b);$$

$$8\,a^5 - 16\,a^4b - 24\,a^3b^2 \equiv 8 \cdot a^3(a+b)(a-3\,b).$$

$$\therefore \text{ H. C. F.} \equiv a(a-3\,b).$$

Hence, to obtain the H. C. F. of two or more expressions, *we find the product of their common factors, each to the lowest power to which it occurs in any of them.*

Exercise 64.

Find the H. C. F. of each of the following expressions:

1. ab^3, a^2b.
2. a^4b^3, a^3b, ab^3.
3. a^2bx^2, ab^2x^2, a^2b^2x.
4. $3\,a^4$, $2\,a^3$, $4\,a^5$, a^2.
5. $10\,x^2$, $15\,x^3$, 5.
6. $10\,x^2y^4z$, $20\,xy^3z^2$, $30\,x^2y^3$.
7. $3\,x^2yz^3$, $15\,xy^3z^2$, $10\,x^2y^2$.
8. $35\,a^2y^3$, $20\,a^2y^4$, $15\,x^2y^3a$.
9. x^2y^n, x^3y^{n-1}, xy^{n+1}.
10. x^ny^{m-1}, $x^{n-1}y^{m+1}$, $x^{n+1}y^m$.
11. $x^3 + y^3$, $x^2 - y^2$.
12. $a^3 - 27$, $9 - a^2$.
13. $a^4 - y^4$, $(a^2 + y^2)^2$.
14. $a^5 - b^5$, $ax - bx$.
15. $x^3 - 1$, $x^2 - 1$.
16. $a^3 + 8$, $a^2 - a - 6$.
17. $a^2 + ab$, $a^3 + b^3$.
18. $x^2 + 3\,x + 2$, $x^2 + 6\,x + 8$.
19. $x^3 + 1$, $x^3 + ax^2 + ax + 1$.
20. $x^4 + 7\,x^2 + 12$, $x^4 + 6\,x^2 + 8$.
21. $x^3 + 3\,x^2y + 2\,xy^2$, $x^4 + 6\,x^3y + 8\,x^2y^2$.

22. $3a^2 - 4ab + b^2,\ 4a^4 - 5a^3b + a^2b^2.$

23. $a^3 - a^2x,\ a^3 - ax^2,\ a^4 - ax^3.$

24. $x^2 - 1,\ x^3 + 1,\ x^2 - 2x - 3.$

25. $2x^2 - 7x + 3,\ 3x^2 - 7x - 6,\ 4x^2 - 17x + 15.$

26. $12x^2 + x - 1,\ 15x^2 + 8x + 1,\ 6x^2 + 11x + 3.$

27. $2x^2 + 9x + 4,\ 2x^2 + 11x + 5,\ 2x^2 - 3x - 2.$

28. $a^3x - a^2bx - 6ab^2x,\ a^2bx^2 - 4ab^2x^2 + 3b^3x^2.$

29. $x^5 - xy^2,\ x^3 + x^2y + xy + y^2,\ x^6 - y^6.$

154. **The polynomial factor of the H. C. F.** of two expressions can always be found by a process analogous to that employed in arithmetic to find the G. C. M. of two numbers.

This process depends upon the two following principles:

(i) *If one integral expression is exactly divisible by another, the second is the H. C. F. of the two expressions.*

E.g., $(x^3 - y^3) \div (x^2 + xy + y^2) \equiv x - y$; hence by definition $x^2 + xy + y^2$ is the H. C. F. of $x^3 - y^3$ and $x^2 + xy + y^2$.

(ii) *If one integral expression is divided by another* (of the same or lower degree in the letter of arrangement), *and if there is a remainder, the H. C. F. of this remainder and the divisor is the H. C. F. of the first two expressions.*

E.g., the remainder obtained by dividing the expression

$$x^3 - 2x^2 - 5x + 6, \text{ or } (x-1)(x+2)(x-3), \qquad (1)$$

by

$$x^2 - 3x + 2, \text{ or } (x-1)(x-2), \qquad (2)$$

is

$$-4x + 4, \text{ or } -4(x-1). \qquad (3)$$

The H. C. F. of the remainder (3) and the divisor (2) is evidently the same as the H. C. F. of the two expressions (1) and (2).

Proof of (ii). Let A and B denote any two integral literal expressions arranged in descending powers of some common letter, the degree of B not being higher than that of A.

Let Q be the quotient and R the remainder obtained by dividing A by B;

then $$A = BQ + R. \tag{1}$$

From (1) $$R = A - BQ. \tag{2}$$

Every factor common to B and R is by § 136 a factor of $BQ + R$, or A; hence every factor common to B and R is common to A and B.

Again, every factor common to A and B is by § 136 a factor of $A - BQ$, or R; hence every factor common to A and B is common to B and R.

Hence, the H. C. F. of B and R is the H. C. F. of A and B.

The following example will illustrate the use of principles (i) and (ii) in finding the H. C. F. of two expressions:

Ex. 1. Find the H. C. F. of $x^2 + x^3 - 2$ and $x^3 + 2x^2 - 3$.

Dividing $x^3 + 2x^2 - 3$ by $x^3 + x^2 - 2$ we obtain the remainder $x^2 - 1$.

Hence, by (ii), the H. C. F of the remainder $x^2 - 1$ and the divisor $x^3 + x^2 - 2$ is the H. C. F. of the two given expressions.

Dividing $x^3 + x^2 - 2$ by $x^2 - 1$ we obtain the second remainder $x - 1$. Hence, by (ii), the H. C. F. of the second remainder $x - 1$, and the second divisor $x^2 - 1$ is the H. C. F. of $x^3 + x^2 - 2$ and $x^2 - 1$, and therefore the H. C. F. of the two given expressions.

But $x^2 - 1$ is exactly divisible by $x - 1$; hence, by (i), $x - 1$ is the H. C. F. of $x^2 - 1$ and $x - 1$, and therefore, by (ii), of the two given expressions.

The work can be arranged as below:

$$\begin{array}{l}
x^3 + x^2 - 2)\,x^3 + 2x^2 - 3\,(1 \\
\qquad\qquad\quad \underline{x^3 + x^2 - 2} \\
\qquad\qquad\qquad\quad x^2 - 1)\,x^3 + x^2 - 2\,(x + 1 \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{x^3 \phantom{{}+x^2} - x\phantom{{}-2}} \\
\qquad\qquad\qquad\qquad\qquad\qquad x^2 + x - 2 \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{x^2 \phantom{{}+x} - 1} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad x - 1)\,x^2 - 1\,(x + 1
\end{array}$$

Before employing the method given above, *all monomial* factors should be removed from the given polynomials, and the H. C. F. of these monomial factors found by factoring.

Ex. 2. Find the H. C. F. of

$$3\,c^2x^4 + 3\,c^2x^3 - 6\,c^2x \text{ and } 6\,cx^5 + 12\,cx^4 - 18\,cx^2.$$

$$3\,c^2x^4 + 3\,c^2x^3 - 6\,c^2x \equiv 3\,c^2x\,(x^3 + x^2 - 2),$$

and $$6\,cx^5 + 12\,cx^4 - 18\,cx^2 \equiv 6\,cx^2\,(x^3 + 2\,x^2 - 3).$$

The H. C. F. of the monomial factors is $3\,cx$; and by example 1, the H. C. F. of the trinomial factors is $x - 1$.

Hence the H. C. F. of the given expressions is $3\,cx\,(x - 1)$.

155. *The H. C. F. of two expressions will not be changed if either expression is multiplied or divided by a factor which is not a factor of the other expression.*

Proof. The factor introduced, by multiplication, into one expression is not a factor of the other expression, and therefore will not be a factor of their H. C. F.

In like manner, the factor removed, by division, from one expression is not a factor of the other expression, and therefore would not be a factor of their H. C. F.

The following examples illustrate how this principle frequently simplifies the work of finding the H. C. F. of two expressions.

Ex. 1. Find the H. C. F. of

$$2\,ax^3 + 8\,ax^2 - 16\,ax + 48\,a \text{ and } 4\,a^2x^4 - 4\,a^2x^3 + 32\,a^2x - 32\,a^2.$$

$$4\,a^2x^4 - 4\,a^2x^3 + 32\,a^2x - 32\,a^2 \equiv 4\,a^2\,(x^4 - x^3 + 8\,x - 8),$$

and $$2\,ax^3 + 8\,ax^2 - 16\,ax + 48\,a \equiv 2\,a\,(x^3 + 4\,x^2 - 8\,x + 24).$$

The H. C. F. of the monomial factors is $2\,a$.

To find the H. C. F. of the polynomial factors we arrange each expression in descending powers of x and proceed as in § 154.

$$\begin{array}{l}
x^3 + 4\,x^2 - 8\,x + 24)\; x^4 - x^3 + 8\,x - 8\;(x - 5 \\
\qquad\qquad\qquad\qquad\;\; \underline{x^4 + 4\,x^3 - 8\,x^2 + 24\,x} \\
\qquad\qquad\qquad\qquad\qquad - 5\,x^3 + 8\,x^2 - 16\,x - 8 \\
\qquad\qquad\qquad\qquad\qquad \underline{- 5\,x^3 - 20\,x^2 + 40\,x - 120} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad 28\,x^2 - 56\,x + 112 = 28\,(x^2 - 2\,x + 4)
\end{array}$$

By § 155 we reject the monomial factor 28, and continue the process with $x^2 - 2x + 4$ as the second divisor.

$$
\begin{array}{r}
x^2 - 2x + 4)\, x^3 + 4x^2 - 8x + 24\,(x + 6 \\
\underline{x^3 - 2x^2 + 4x \qquad\quad} \\
6x^2 - 12x + 24 \\
\underline{6x^2 - 12x + 24}
\end{array}
$$

By (i), $x^2 - 2x + 4$ is the H. C. F. of the first divisor and the first remainder, and hence, by (ii), of the polynomial factors.

Therefore the H. C. F. sought is $2a(x^2 - 2x + 4)$.

Ex. 2. Find the H. C. F. of

$$2x^2 - 5x + 2 \text{ and } x^3 + 4x^2 - 4x - 16.$$

Since 2 is not a factor of $2x^2 - 5x + 2$, we can by § 155 multiply $x^3 + 4x^2 - 4x - 16$ by 2, and thus avoid the inconvenience of fractions.

The work may be written as below :

$$
\begin{array}{lr}
 & 2x^2 - 5x + 2)\, x^3 + 4x^2 - 4x - 16 \\
\text{Multiply by 2,} & \underline{2 \qquad\qquad\qquad\qquad} \\
 & 2x^3 + 8x^2 - 8x - 32\,(x \\
 & \underline{2x^3 - 5x^2 + 2x \qquad\quad} \\
 & 13x^2 - 10x - 32 \\
\text{Multiply by 2,} & 26x^2 - 20x - 64\,(13 \\
 & \underline{26x^2 - 65x + 26} \\
\text{Divide by 45,} & 45)\,\underline{45x - 90} \\
 & x - 2)\, 2x^2 - 5x + 2\,(2x - 1
\end{array}
$$

Hence the H. C. F. is $x - 2$.

Ex. 3. Find the H. C. F. of

$$2x^3 + x^2 - x - 2 \text{ and } 3x^3 - 2x^2 + x - 2.$$

Multiply the last expression by 2.

$$
\begin{array}{r}
2x^3 + x^2 - x - 2)\, 6x^3 - 4x^2 + 2x - 4\,(3 \\
\underline{6x^3 + 3x^2 - 3x - 6} \\
-7x^2 + 5x + 2
\end{array}
$$

Multiply the first divisor by 7.

$$
\begin{array}{r|l}
-7x^2+5x+2) & 14x^3+7x^2-7x-14\,(-2x \\
 & 14x^3-10x^2-4x \\ \hline
 & \quad\quad 17x^2-3x-14
\end{array}
$$

Multiply this remainder by 7,

$$
\begin{array}{l}
119x^2-21x-98\,(-17 \\
119x^2-85x-34 \\ \hline
\end{array}
$$

Divide by 64, $\quad 64)\,64x-64$

$$
\begin{array}{r|l}
x-1) & -7x^2+5x+2\,(-7x-2 \\
 & -7x^2+7x \\ \hline
 & \quad\quad -2x+2 \\
 & \quad\quad -2x+2 \\ \hline
\end{array}
$$

Hence the H. C. F. sought is $x-1$.

In the above process of finding the H. C. F. of two integral expressions, each remainder is evidently of a lower degree in the letter of arrangement than the preceding one. Hence unless at some stage of the process the remainder is zero, we must come at last to a remainder which does not contain the letter of arrangement. In this case the given expressions have no common polynomial factor containing that letter; for by § 154 this last remainder contains all the polynomial factors common to the given expressions.

156. By the foregoing principles we have the following rule for finding the H. C. F. of two expressions:

Remove from the given expressions all monomial factors, and set aside their H. C. F. as a factor of the required H. C. F.

Divide the expression of the higher degree arranged in descending powers of the common letter of arrangement by the other expression; if both expressions are of the same degree either can be taken as the first divisor.

Divide the first divisor by the first remainder; the second divisor by the second remainder; and so on, until the last

remainder is zero or does not contain the letter of arrangement.

If the last remainder is zero, the last divisor is the H. C. F. sought; if the last remainder is not zero, the two expressions have no common factor in the letter of arrangement.

Any dividend can be multiplied by any number which is not a factor of the corresponding divisor; or any divisor can be divided by any number which is not a factor of the corresponding dividend.

157. Any factor common to three or more expressions must be a factor of the H. C. F. of any two of them.

Hence, to find the H. C. F. of three expressions, we can first find the H. C. F. of any two of them, and then find the H. C. F. of this result and the third.

Ex. Find the H. C. F. of

$$x^3 + x^2 - x - 1,\ x^3 + 3x^2 - x - 3, \text{ and } x^3 + x^2 - 2.$$

The H. C. F. of the first two expressions is $x^2 - 1$.

The H. C. F. of $x^2 - 1$ and $x^3 + x^2 - 2$ is $x - 1$.

Hence the H. C. F. sought is $x - 1$.

Whenever the given expressions can be factored by inspection, their H. C. F. should always be obtained by factoring.

Exercise 65.

Find the H. C. F. of the following expressions:

1. $x^2 - 5x + 4,\ x^3 - 5x^2 + 4.$
2. $x^2 - 5xy + 4y^2,\ x^4 - 5x^3y + 4xy^3.$
3. $2x^2 - 5x + 2,\ 4x^3 + 12x^2 - x - 3.$
4. $x^3 - 5x^2 - 99x + 40,\ x^3 - 6x^2 - 86x + 35.$
5. $x^3 + 2x^2 - 8x - 16,\ x^3 + 3x^2 - 8x - 24.$
6. $x^3 - x^2 - 5x - 3,\ x^3 - 4x^2 - 11x - 6.$
7. $x^3 + 3x^2 - 8x - 24,\ x^3 + 3x^2 - 3x - 9.$

8. $x^6 + 3x^4y - 10x^2y^2,\ x^4 - 3x^2y + 2y^2.$

9. $2a^2 - 5a + 2,\ 2a^3 - 3a^2 - 8a + 12.$

10. $2b^2 - 5b + 2,\ 12b^3 - 8b^2 - 3b + 2.$

11. $a^3 - 5a^2x + 7ax^2 - 3x^3,\ a^3 - 3ax^2 + 2x^3.$

12. $x^4 - 2x^3 - 4x - 7,\ x^4 + x^3 - 3x^2 - x + 2.$

13. $x^3 - 3a^2x - 2a^3,\ x^3 - ax^2 - 4a^3.$

14. $2x^3 + 4x^2 - 7x - 14,\ 6x^3 - 10x^2 - 21x + 35.$

15. $2x^4 - 2x^3 + x^2 + 3x - 6,\ 4x^4 - 2x^3 + 3x - 9.$

16. $3x^3 + x^2 + x - 2,\ 2x^3 - x^2 - x - 3.$

17. $3x^3 - 3ax^2 + 2a^2x - 2a^3,\ 3x^3 + 12ax^2 + 2a^2x + 8a^3.$

18. $3x^3 - 3x^2y + xy^2 - y^3,\ 4x^2y - 5xy^2 + y^3.$

19. $12x^2 - 15xy + 3y^2,\ 6x^3 - 6x^2y + 2xy^2 - 2y^3.$

20. $10x^3 + 25ax^2 - 5a^3,\ 4x^3 + 9ax^2 - 2a^2x - a^3.$

21. $6a^3 + 13a^2x - 9ax^2 - 10x^3,\ 9a^3 + 12a^2x - 11ax^2 - 10x^3.$

22. $2x^4 + 9x^3 + 14x + 3,\ 2 + 9x + 14x^3 + 3x^4.$

23. $3x^4 + 5x^3 - 7x^2 + 2x + 2,\ 2x^4 + 3x^3 - 2x^2 + 12x + 5.$

24. $2x^3 - 11x^2 + 11x + 4,\ 2x^4 - 3x^3 + 7x^2 - 12x - 4.$

25. $2x^4 + 4x^3 + 3x^2 - 2x - 2,\ 3x^4 + 6x^3 + 7x^2 + 2x + 2.$

26. $x^2 - 9x - 10,\ x^2 - 7x - 30,\ x^2 - 11x + 10.$

27. $x^2 + x - 6,\ x^3 - 2x^2 - x + 2,\ x^3 + 3x^2 - 6x - 8.$

28. $x^3 + 7x^2 + 5x - 1,\ x^2 + 3x - 3x^3 - 1,\ 3x^3 + 5x^2 + x - 1.$

LOWEST COMMON MULTIPLE.

158. A **common multiple** of two or more integral expressions is any integral expression which is exactly divisible by each of them.

The **lowest common multiple** (L. C. M.) of two or more integral literal expressions is the integral expression of *lowest degree*, which is exactly divisible by each of them.

The *numeral* factor of the L. C. M. is the least common multiple (L. C. M.) of the numeral factors of the given expressions.

E.g., a^3b^3 is the L. C. M. of a^3b, ab^3, and a^2b^2.

Again, the L. C. M. of $12\,axy^2z^3$ and $15\,b^3y^4$ is $60\,ab^3xy^4z^3$.

159. L. C. M. by factoring.

Ex. 1. Find the L. C. M. of ab^2, a^3bc^3, and ab^2c^5.

The L. C. M. of these expressions cannot contain a lower power of a than a^3, a lower power of b than b^2, and a lower power of c than c^5.

Hence, the required L. C. M. is $a^3b^2c^5$.

Observe that the power of each base in the L. C. M. is the *highest* power to which it occurs in any of the given expressions.

When the expressions involve numeral factors, the L. C. M. of these factors should be obtained as in Arithmetic.

Ex. 2. Find the L. C. M. of $x^2 + 7\,x + 12$, $x^2 + 6\,x + 8$, and $5\,x^2 + 20\,x + 20$.

$$x^2 + 7\,x + 12 \equiv (x + 3)(x + 4);$$

$$x^2 + 6\,x + 8 \equiv (x + 2)(x + 4);$$

$$5\,x^2 + 20\,x + 20 \equiv 5(x + 2)^2.$$

$$\therefore \text{ L. C. M.} = 5(x + 2)^2(x + 3)(x + 4).$$

These examples illustrate the following rule:

To obtain the L. C. M. of two or more integral expressions, *multiply the L. C. M. of their numeral factors by the product of all their prime literal factors, each to the highest power to which it occurs in any one of them.*

Proof. The L. C. M. by definition contains each factor the greatest number of times that it occurs in any one of the given expressions.

Exercise 66.

Find the L. C. M. of the following expressions:

1. $4x^3y,\ 10xy^3$.
2. $24a^3b^3x^4,\ 60a^2b^4x^6$.
3. $9a^2b^3x^4y^5,\ 8x^3y^6$.
4. $x^2,\ x^2-3x$.
5. $21x^3,\ 7x^2(x+1)$.
6. $6x^2-2x,\ 9x^2-3x$.
7. $x^2+2x,\ x^2+3x+2$.
8. $x^2-5x+4,\ x^2-6x+8$.
9. $x^2+4x+4,\ x^2+5x+6$.
10. $x^2-x-6,\ x^2+x-2,\ x^2-4x+3$.
11. $x^2+x-20,\ x^2-10x+24,\ x^2-x-30$.
12. $x^2+x-42,\ x^2-11x+30,\ x^2+2x-35$.
13. $2x^2+3x+1,\ 2x^2+5x+2,\ x^2+3x+2$.
14. $5x^2+11x+2,\ 5x^2+16x+3,\ x^2+5x+6$.
15. $x^2-7xy+12y^2,\ x^2-6xy+8y^2,\ x^2-5xy+6y^2$.
16. $2x^2+3x-2,\ 2x^2+15x-8,\ x^2+10x+16$.
17. $8x^2-38xy+35y^2,\ 4x^2-xy-5y^2,\ 2x^2-5xy-7y^2$.

160. L. C. M. by H. C. F. The L. C. M. of two expressions can always be obtained by first finding their H. C. F.

Ex. 1. Find the L. C. M. of x^3+x^2-2 and x^3+2x^2-3.

The H. C. F. of these expressions is found to be $x-1$.

By division we find that

$$x^3+x^2-2 \equiv (x-1)(x^2+2x+2),$$

and
$$x^3+2x^2-3 \equiv (x-1)(x^2+3x+3).$$

Since $x-1$ is the H. C. F. of the given expressions, their second factors x^2+2x+2 and x^2+3x+2 have no common factor.

Hence, the required L. C. M. is

$$(x-1)(x^2+2x+2)(x^2+3x+3). \qquad (1)$$

161. To find the L. C. M. of *three* expressions A, B, C, we find M, the L. C. M. of A and B; then the L. C. M. of M and C is the L. C. M. required.

Exercise 67.

Find the H. C. F. and L. C. M. of:

1. $2x^2 + 3x - 20,\ 6x^3 - 25x^2 + 21x + 10.$
2. $x^2 - 15x + 36,\ x^3 - 3x^2 - 2x + 6.$
3. $9x^3 - x - 2,\ 3x^3 - 10x^2 - 7x - 4.$
4. $x^3 + x^2 - 4x - 4,\ x^3 + 6x^2 + 11x + 6.$
5. $x^3 - x^2 - 7x + 15,\ x^3 + x^2 - 3x + 9.$
6. $x^3 - x^2 + x + 3,\ x^4 + x^3 - 3x^2 - x + 2.$
7. $x^4 - x^3 + 8x - 8,\ x^3 + 4x^2 - 8x + 24.$
8. $6x^3 + x^2 - 5x - 2,\ 6x^3 + 5x^2 - 3x - 2.$
9. $4x^3 - 10x^2 + 4x + 2,\ 3x^4 - 2x^3 - 3x + 2.$
10. $x^3 - 9x^2 + 26x - 24,\ x^3 - 12x^2 + 47x - 60.$
11. $x^3 - ax^2 - a^2x + a^3,\ x^3 + ax^2 - a^2x - a^3.$

Find the L. C. M. of:

12. $x^3 - 6x^2 + 11x - 6,\ x^3 - 9x^2 + 26 - 24,$
$$x^3 - 8x^2 + 19x - 12.$$
13. $x^3 - 5x^2 + 9x - 9,\ x^3 - x^2 - 9x + 9,$
$$x^4 - 4x^2 + 12x - 9.$$

162. *The L. C. M. of two integral expressions is the product of either expression into the quotient of the other divided by the H. C. F. of the two expressions.*

Proof. Let A and B denote any two integral expressions, H their H. C. F., and L their L. C. M.

Then L by definition contains all the factors of A, and in addition all the factors of B which are not in A; that is, the factors of $B \div H$.

Hence $$L \equiv A \times (B \div H). \tag{1}$$

163. From (1) in § 162 we obtain

$$A \times B \equiv L \times H. \tag{2}$$

That is, *the product of two integral expressions is equal to the product of their* L. C. M. and their H. C. F.

CHAPTER XII

FRACTIONS

164. A fraction being an indicated quotient, the fraction a/b denotes that number which multiplied by the divisor b is equal to the dividend a. Reread § 90.

E.g., $-8/4=-2$, for $-2\times 4=-8$.

165. Algebraic fractions. The fractions in arithmetic involve only arithmetic numbers, and are called *arithmetic* fractions.

In Chapter III. we used arithmetic fractions to denote the arithmetic values of positive and negative numbers, the quality being indicated by the sign $^+$ or $^-$.

An *algebraic fraction* is one whose numerator and denominator are quality-numbers. The sign before an algebraic fraction denotes the quality of its numeral coefficient. Thus, $-\dfrac{-4}{+3}$ denotes the product of -1 and the fraction $(-4)/(+3)$.

166. By the law of quality in division it follows that —

Changing the quality of both the numerator and denominator does not change the quality of the fraction.

Changing the quality of either the numerator or denominator changes the quality of the fraction.

E.g., $$\frac{-7}{-8}=\frac{7}{8}, \text{ and } \frac{-a}{b}\equiv\frac{a}{-b}.$$

But $\dfrac{-a}{b}$ and $\dfrac{-a}{-b}$ are opposite in quality.

167. By § 166 and the law of quality in § 48 it follows that —

Changing the sign before a fraction and changing the quality of either its numerator or denominator does not change the quality of the term.

E.g., $-\frac{-8}{9}=\frac{8}{9};\quad \frac{a}{b}\equiv-\frac{-a}{b}$ or $-\frac{a}{-b}$.

Again, $\frac{abc}{xyz}\equiv\frac{(-a)(-b)(-c)}{xyz};\quad \frac{-abc}{xyz}\equiv-\frac{-abc}{(-x)yz}$.

168. A **fractional literal expression** is an expression which has one or more fractional literal terms.

E.g., $\frac{a+b}{x-y}$ and $ax+by+\frac{x}{a+b}$ are fractional expressions.

An *integral* literal expression, as we have seen, denotes any *integral* or *fractional* number; likewise a *fractional* literal expression denotes any *integral* or *fractional* number.

E.g., a denotes 2, 5, 3/2, $-2/3$, or any other number.

Again, when $a=6$ and $b=2$, $a/b=3$;

when $a=12$ and $b=3$, $a/b=4$;

when $a=3$ and $b=5$, $a/b=3/5$; and so on.

169. A **proper literal fraction** is a fraction whose numerator is of a lower degree than its denominator in a common letter of arrangement.

An **improper literal fraction** is a fraction whose numerator is of the same or of a higher degree than its denominator in a common letter of arrangement.

E.g., $\frac{1}{x+2}$ and $\frac{x+1}{x^2+3x-4}$ are *proper* literal fractions.

While $\frac{x^2}{x^2+1}$ and $\frac{x^3+1}{x^2+3x-4}$ are *improper* literal fractions.

The value of a proper literal fraction may be either *less* or *greater* than 1.

REDUCTION OF FRACTIONS.

170. To **reduce** an expression is to find an *identical* expression of some required form.

171. To reduce an improper fraction to an expression containing no improper fractions,

Perform the indicated operation of division.

Sometimes the quotient can be obtained by inspection.

For examples, see §§ 129 and 133.

Exercise 68.

Reduce each of the following improper fractions to fractional expressions containing no improper fractions:

1. $\dfrac{5x^2-20x-15}{5x}$. 5. $\dfrac{x^2+a^2}{x+a}$. 9. $\dfrac{x^4+a^4}{x-a}$.

2. $\dfrac{x+7}{x+2}$. 6. $\dfrac{x^2+a^2}{x-a}$. 10. $\dfrac{2x^2-7x-1}{x-3}$.

3. $\dfrac{x^2}{x+3}$. 7. $\dfrac{x^3-a^3}{x+a}$. 11. $\dfrac{x^2-3x}{x-2}$.

4. $\dfrac{5x+7}{x-3}$. 8. $\dfrac{x^4+16}{x+2}$. 12. $\dfrac{3x^2+2x+1}{x+4}$.

13. $\dfrac{4a^2+6ab+9b^2}{2a-3b}$. 14. $\dfrac{60x^3-17x^2-4x+1}{5x^2+9x-2}$.

172. *The value of a fraction is not changed by multiplying its numerator and denominator by the same number.*

That is, $$a/b \equiv am/(bm).$$

Proof. $\dfrac{a}{b} \equiv \dfrac{a}{b} \times \dfrac{m}{m}$, m/m being a form of 1

$\equiv am/(bm)$. § 91

E.g., $\dfrac{3}{4} = \dfrac{3 \times 5}{4 \times 5} = \dfrac{15}{20}$, $\dfrac{x-y}{x+y} \equiv \dfrac{(x-y)(x+y)}{(x+y)(x+y)} \equiv \dfrac{x^2-y^2}{(x+y)^2}$.

173. *The value of a fraction is not changed by dividing its numerator and denominator by the same number.*

That is, $\boldsymbol{a/b \equiv (a \div m)/(b \div m)}$.

Proof. $$\frac{a \div m}{b \div m} \equiv \frac{(a \div m)m}{(b \div m)m} \equiv \frac{a}{b}. \qquad \S\ 172$$

E.g., $$\frac{c + cx}{c + cy} \equiv \frac{(c + cx) \div c}{(c + cy) \div c} \equiv \frac{1 + x}{1 + y}.$$

174. A fraction is said to be in its lowest terms when its numerator and denominator have no common factor.

175. To reduce a fraction to its lowest terms,

Divide its numerator and denominator by all their common factors, or by their H. C. F. (§ 173).

Ex. 1. Reduce $\dfrac{4\,ax^3y^2}{8\,a^2xy^4}$ to its lowest terms.

The H. C. F. of the numerator and denominator is $4\,axy^2$; and

$$\frac{4\,ax^3y^2}{8\,a^2xy^4} \equiv \frac{4\,ax^3y^2 \div 4\,axy^2}{8\,a^2xy^4 \div 4\,axy^2} \equiv \frac{x^2}{2\,ay^2}. \qquad \S\ 173$$

Ex. 2. Reduce $\dfrac{a^2 - ax}{a^2 - x^2}$ to its lowest terms.

Factoring numerator and denominator, we obtain

$$\frac{a^2 - ax}{a^2 - x^2} \equiv \frac{a(a - x)}{(a + x)(a - x)} \equiv \frac{a}{a + x}. \qquad \S\ 173$$

Ex. 3. $$\frac{x^4 - 1}{x^6 - x^5 - x^2 + x} \equiv \frac{x^4 - 1}{x^2(x^4 - 1) - x(x^4 - 1)}$$
$$\equiv \frac{1}{x^2 - x} \equiv \frac{1}{x(x - 1)}.$$

Exercise 69.

Reduce to its lowest terms each of the following fractions:

1. $\dfrac{-x^3y}{-x^2y^3}$.

2. $-\dfrac{-2\,a^2bc^2}{4\,a^4b^3c}$.

3. $-\dfrac{x^4y^6z^2}{-x^6y^2z^2}$.

4. $\dfrac{12\,x^7y^2z^{10}}{16\,x^2y^2z^2}$.

5. $\dfrac{-15\,a^3b^2c^7x^5}{-25\,a^2b^4c^3x^8}$.

6. $\dfrac{125\,ab^2c^3d^4}{150\,a^4b^3c^2d}$.

7. $\dfrac{3\,a^5x^3yz^4}{-5\,ab^4xy^3z}$.

8. $\dfrac{5\,a^3b^4c^5xy^2}{-7\,b^4c^3xy^3}$.

9. $\dfrac{3\,a^3bc^4x^5y^2z}{4\,a^2b^3cx^3y^4z^3}$.

10. $\dfrac{a^2-ab}{a^2+ab}$.

11. $\dfrac{x^2+ax}{x^2-a^2}$.

12. $\dfrac{x^2-x^2y^2}{(x+xy)^2}$.

13. $\dfrac{x^2+2\,x}{x^2-4}$.

14. $\dfrac{4\,x-16}{x^2-16}$.

15. $\dfrac{2\,x^3-4\,x^4}{x^2-4\,x^4}$.

16. $\dfrac{x-2}{4-x^2}$.

17. $\dfrac{a-3}{9-a^2}$.

18. $\dfrac{x^4-a^2}{a^2-ax^2}$.

19. $\dfrac{x^2-1}{x^3-1}$.

20. $\dfrac{x^4-a^4}{x^6-a^6}$.

21. $\dfrac{15\,a^2-5\,ax}{x^2-9\,a^2}$.

22. $\dfrac{a^2-2\,ax+x^2}{x^2-a^2}$.

23. $\dfrac{a^4+2\,a^2b^2+b^4}{b^4-a^4}$.

24. $\dfrac{1-5\,a+6\,a^2}{1-7\,a+12\,a^2}$.

25. $\dfrac{x^2-9\,x+20}{x^2+6\,x-55}$.

26. $\dfrac{1-9\,y^2+20\,y^4}{1+6\,y^2-55\,y^4}$.

27. $\dfrac{x^4+x^2-2}{x^4+5\,x^2+6}$.

28. $\dfrac{x^{2n}+2\,x^n+1}{x^{2n}+3\,x^n+2}$.

29. $\dfrac{(x^3-y^3)(x^2-xy+y^2)}{(x^3+y^3)(x^2+xy+y^2)}$.

30. $\dfrac{x^3-ax^2+b^2x-ab^2}{x^3-ax^2-b^2x+ab^2}$.

176. When the factors of the numerator and denominator of a fraction cannot be found by inspection, their H. C. F. can be found by the method given in Chapter XI.

Ex. 1. Reduce $\dfrac{3x^3 - 13x^2 + 23x - 21}{15x^3 - 38x^2 - 2x + 21}$ to its lowest terms.

The H. C. F. is found to be $3x - 7$; and by division we find

$$3x^3 - 13x^2 + 23x - 21 \equiv (x^2 - 2x + 3)(3x - 7)$$

$$15x^3 - 38x^2 - 2x + 21 \equiv (5x^2 - x - 3)(3x - 7).$$

$$\therefore \frac{3x^3 - 13x^2 + 23x - 21}{15x^3 - 38x^2 - 2x + 21} \equiv \frac{x^2 - 2x + 3}{5x^2 - x - 3}.$$

Exercise 70.

Reduce to the simplest form the following fractions:

1. $\dfrac{a^3 - 3a + 2}{2a^3 - 3a^2 + 1}$.

2. $\dfrac{a^3 + 3a^2 - 20}{a^4 - a^2 - 12}$.

3. $\dfrac{4x^3 + 3ax^2 + a^3}{x^4 + ax^3 + a^3x + a^4}$.

4. $\dfrac{4x^3 - 10x^2 + 4x + 2}{3x^4 - 2x^3 - 3x + 2}$.

5. $\dfrac{6x^3 + x^2 - 5x - 2}{6x^3 + 5x^2 - 3x - 2}$.

6. $\dfrac{2x^3 + ax^2 + 4a^2x - 7a^3}{x^3 - 7ax^2 + 8a^2x - 2a^3}$.

7. $\dfrac{2x^3 + 3x^2 + 4x - 3}{6x^3 + x^2 - 1}$.

8. $\dfrac{x^4 - x^3 - x + 1}{x^4 - 2x^3 - x^2 - 2x + 1}$.

9. $\dfrac{4x^4 + 11x^2 + 25}{4x^4 - 9x^2 + 30x - 25}$.

10. $\dfrac{x^4 - 20x^2 - 15x + 4}{x^4 + 9x^3 + 19x^2 - 9x - 20}$.

177. Two or more fractions which have the same denominator are said to have a common denominator.

The **lowest common denominator** (L. C. D.) of two or more fractions is the L. C. M. of their denominators.

E.g., the L. C. D. of the fractions $\dfrac{a}{a^2 - b^2}$ and $\dfrac{3x}{(a - b)^2}$ is $(a - b)^2(a + b)$, or the L. C. M. of the denominators $a^2 - b^2$ and $(a - b)^2$.

178. To reduce two or more fractions to identical fractions having the L. C. D.,

Multiply both the numerator and the denominator of each fraction by the quotient obtained by dividing their L. C. D. by the denominator of that fraction.

Proof. The derived fractions have the L. C. D.; and by § 172, each is identical with its corresponding given fraction.

Ex. 1. Reduce $\dfrac{x}{a^2b(x+a)}$, $\dfrac{y}{ab^2(x-a)}$, and $\dfrac{z}{ab(x^2-a^2)}$ to identical fractions having the L. C. D.

The L. C. M. of the denominators is $a^2b^2(x^2-a^2)$.

Dividing this L. C. D. by the denominator of each fraction, and multiplying both the numerator and denominator by the quotient, we obtain

$$\frac{x}{a^2b(x+a)} \equiv \frac{x \times b(x-a)}{a^2b^2(x^2-a^2)} \equiv \frac{bx(x-a)}{a^2b^2(x^2-a^2)},$$

$$\frac{y}{ab^2(x-a)} \equiv \frac{y \times a(x+a)}{a^2b^2(x^2-a^2)} \equiv \frac{ay(x+a)}{a^2b^2(x^2-a^2)},$$

and
$$\frac{z}{ab(x^2-a^2)} \equiv \frac{z \times ab}{a^2b^2(x^2-a^2)} \equiv \frac{abz}{a^2b^2(x^2-a^2)}.$$

Ex. 2. Reduce $\dfrac{1}{x^2-5x+6}$, $\dfrac{1}{x^2-4x+3}$, $\dfrac{1}{x^2-3x+2}$ to identical fractions having the L. C. D.

The denominators equal

$$(x-3)(x-2),\ (x-3)(x-1),\ (x-2)(x-1),$$

respectively. Hence their L. C. M. is $(x-3)(x-2)(x-1)$.

$$\therefore\ \frac{1}{(x-3)(x-2)} \equiv \frac{x-1}{(x-3)(x-2)(x-1)};$$

$$\frac{1}{(x-3)(x-1)} \equiv \frac{x-2}{(x-3)(x-2)(x-1)};$$

$$\frac{1}{(x-2)(x-1)} \equiv \frac{x-3}{(x-3)(x-2)(x-1)}.$$

Exercise 71.

Reduce to identical fractions having the L. C. D.:

1. $\dfrac{3}{4x}, \dfrac{4}{6x^2}, \dfrac{5}{12x^3}.$

2. $\dfrac{5a^2}{6x^2y}, \dfrac{3bx}{8y^2x}, \dfrac{7cy-m}{10xz^2}.$

3. $\dfrac{a}{x-a}, \dfrac{x}{x-a}, \dfrac{a^2}{x^2-a^2}.$

4. $\dfrac{2}{a-b}, \dfrac{3}{a+b}, \dfrac{4}{a^2+b^2}.$

5. $\dfrac{ay}{1-x}, \dfrac{ax^2}{(1-x)^2}, \dfrac{xy^3}{(1-x)^3}.$

6. $\dfrac{m}{n}, \dfrac{m^3}{mn-n^2}, \dfrac{mn^2}{m^2-n^2}.$

7. $\dfrac{ab}{am-bm+an-bn}, \dfrac{m-n}{2a^2-2ab}.$

8. $\dfrac{3}{x^2+3x+2}, \dfrac{5}{x^2+2x-3}, \dfrac{2}{x^2+5x+6}.$

9. $\dfrac{2a}{a-b}, \dfrac{b}{2b-2a}, \dfrac{3a^2}{4(a^2-b^2)}, \dfrac{5b^2}{6(b^2-a^2)}.$

10. $\dfrac{1}{(x-a)(x-b)}, \dfrac{1}{(b-x)(c-x)}, \dfrac{1}{(x-c)(x-a)}.$

ADDITION AND SUBTRACTION OF FRACTIONS.

179. The converse of the distributive law for division is

$$\frac{a}{x}+\frac{b}{x}-\frac{c}{x}\equiv\frac{a+b-c}{x}.$$

Hence to add or subtract fractions,

Reduce the fractions, if they have not a C. D., to identical fractions having the L. C. D.; then add or subtract each numerator as the sign before the fraction directs, and write the result over the L. C. D.

E.g., $$\frac{b}{x}+\frac{c}{y}\equiv\frac{by}{xy}+\frac{cx}{xy}\equiv\frac{by+cx}{xy};$$

and $$\frac{a}{c}-\frac{b}{x}\equiv\frac{ax}{cx}-\frac{bc}{cx}\equiv\frac{ax-bc}{cx}.$$

NOTE. The student should remember that when either the numerator or the denominator is a polynomial, the horizontal line in a fraction is a sign of grouping as well as a sign of division.

Ex. 1. Combine and simplify $\dfrac{1}{x-y}+\dfrac{1}{x+y}$.

The L. C. M. of the denominators is $(x-y)(x+y)$; and

$$\frac{1}{x-y}+\frac{1}{x+y}\equiv\frac{x+y}{(x-y)(x+y)}+\frac{x-y}{(x-y)(x+y)}$$
$$\equiv\frac{x+y+(x-y)}{x^2-y^2}=\frac{2x}{x^2-y^2}.$$

Ex. 2. Combine and simplify $\dfrac{1}{x^2-5x+6}-\dfrac{1}{x^2-7x+12}$.

$$x^2-5x+\ \ 6\equiv(x-2)(x-3),$$
$$x^2-7x+12\equiv(x-3)(x-4);$$

hence, the L. C. M. of the denominators is $(x-2)(x-3)(x-4)$, and

$$\text{the expression }\equiv\frac{x-4}{(x-2)(x-3)(x-4)}-\frac{x-2}{(x-2)(x-3)(x-4)}$$
$$\equiv\frac{(x-4)-(x-2)}{(x-2)(x-3)(x-4)}\equiv\frac{-2}{(x-2)(x-3)(x-4)}.$$

Ex. 3. Combine and simplify $\dfrac{a^2-bc}{bc}-\dfrac{ac-b^2}{ca}-\dfrac{ab-c^2}{ab}$.

The L. C. M. of the denominators is abc, hence

$$\text{the expression }\equiv\frac{a(a^2-bc)}{abc}-\frac{b(ac-b^2)}{abc}-\frac{c(ab-c^2)}{abc}$$
$$\equiv\frac{a(a^2-bc)-b(ac-b^2)-c(ab-c^2)}{abc}$$
$$\equiv\frac{a^3+b^3+c^3-3abc}{abc}.$$

Exercise 72.

Combine and simplify:

1. $\dfrac{a-5b}{5}-\dfrac{a-3b}{3}$.

2. $\dfrac{a-3b}{4}+\dfrac{3a-b}{5}$

3. $\frac{6a-5b}{3}-\frac{4a-7b}{2}$.

4. $\frac{a-b}{a^2b}+\frac{a-b}{ab^2}$.

5. $\frac{x}{4}-\frac{x-4}{3}+\frac{x-5}{6}$.

6. $2\frac{x-y}{3}-3\frac{x+y}{4}$.

7. $5\frac{x-2y}{6}-3\frac{y-2x}{4}$.

8. $5\frac{a-b}{c}-7\frac{a-b}{x}$.

9. $\frac{2x-3}{9}-\frac{x+2}{6}+\frac{5x+8}{12}$.

10. $\frac{a-2b}{2a}-\frac{a-5b}{4a}+\frac{a+7b}{8a}$.

11. $\frac{b+c}{2a}+\frac{c+a}{4b}-\frac{a-b}{3c}$.

12. $\frac{a-x}{x}+\frac{a+x}{a}-\frac{a^2-x^2}{2ax}$.

13. $\frac{2a^2-b^2}{a^2}-\frac{b^2-c^2}{b^2}-\frac{c^2-a^2}{c^2}$.

14. $\frac{2x-3y}{xy}+\frac{3x-2z}{xz}+\frac{5}{x}$.

15. $\frac{x-3}{5x}+\frac{x^2-9}{10x^2}-\frac{8-x^3}{15x^3}$.

16. $\frac{2}{xy}-\frac{3y^2-x^2}{xy^3}+\frac{xy+y^2}{x^2y^2}$.

Reduce to an improper fraction:

17. $a+b-\frac{a^2+b^2}{a-b}$.

$$\frac{a+b}{1}-\frac{a^2+b^2}{a-b}\equiv\frac{a^2-b^2-(a^2+b^2)}{a-b}$$

$$\equiv\frac{-2b^2}{a-b}\equiv\frac{2b^2}{b-a}.$$

18. $a-1+\frac{2}{a+1}$.

19. $a+x+\frac{x^2}{a-x}$.

20. $x+2y+\frac{4y^2}{x-2y}$.

21. $x^2-3x-\frac{3x(3-x)}{x-2}$.

22. $a^2 - 2ax + 4x^2 - \dfrac{6x^3}{a + 2x}$.

23. $x - a + y + \dfrac{a^2 - ay + y^2}{x + a}$.

24. $1 + x + x^2 + x^3 + \dfrac{x^4}{1 - x}$.

25. $\dfrac{x^4 + 2x^2 + 1}{x^2 + x + 1} - (x - x^2 - 1)$.

26. $1 + 2x + 4x^2 + \dfrac{x^3 + 1}{2x - 1}$.

27. $x^2 - 2x + 3 - \dfrac{x^3 + 13x - 5}{x^2 + 3x - 2}$.

28. Combine and simplify $\dfrac{a}{a - x} + \dfrac{ax}{x^2 - a^2}$.

Beginners should always see to it that the denominators of the fractions to be added or subtracted are all arranged in *descending* powers, or all in *ascending* powers, of some particular letter of arrangement.

Arranging the denominators in this example in *descending* powers of a, we have

$$\frac{a}{a - x} + \frac{ax}{x^2 - a^2} \equiv \frac{a}{a - x} + \frac{-ax}{a^2 - x^2}$$
$$\equiv \frac{a(a + x) - ax}{a^2 - x^2} = \frac{a^2}{a^2 - x^2}.$$

Combine and simplify:

29. $\dfrac{a}{a - b} + \dfrac{b}{b - a}$.

30. $\dfrac{x}{x - a} + \dfrac{a}{a - x}$.

31. $\dfrac{x}{x^2 - a^2} + \dfrac{a}{a^2 - x^2}$.

32. $\dfrac{1}{1 - x} - \dfrac{2}{1 - x^2}$.

33. $\dfrac{1}{3 + x} + \dfrac{6}{x^2 - 9}$.

34. $\dfrac{1 + x}{1 - x} - \dfrac{1 - x}{1 + x}$.

35. $\dfrac{1}{x - 2} + \dfrac{2}{(x - 2)^2}$.

36. $\dfrac{a + 2b}{a - 2b} - \dfrac{a - 2b}{a + 2b}$.

37. $\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}+\dfrac{4y^2}{x^2-y^2}.$

38. $\dfrac{2a}{a+b}+\dfrac{2b}{a-b}-\dfrac{a^2+b^2}{a^2-b^2}.$

39. $\dfrac{a}{a-1}-1-\dfrac{1}{a(a-1)}.$

40. $\dfrac{4a^2+b^2}{4a^2-b^2}-\dfrac{2a-b}{2a+b}.$

41. $\dfrac{1}{x-1}+\dfrac{1}{x-2}-\dfrac{1}{x+2}-\dfrac{1}{x+1}.$

The character of the denominators in this example suggests that it is simpler first to combine the first and fourth fractions, next the second and third, and then to combine these results, as below:

$$\frac{1}{x-1}-\frac{1}{x+1}\equiv\frac{x+1-(x-1)}{x^2-1}\equiv\frac{2}{x^2-1},$$

$$\frac{1}{x-2}-\frac{1}{x+2}\equiv\frac{x+2-(x-2)}{x^2-4}\equiv\frac{4}{x^2-4},$$

and $$\frac{2}{x^2-1}+\frac{4}{x^2-4}\equiv\frac{2(x^2-4)+4(x^2-1)}{(x^2-1)(x^2-4)}\equiv\frac{6x^2-12}{x^4-5x^2+4}.$$

42. $\dfrac{1}{1-x}+\dfrac{1}{1+x}+\dfrac{2}{1+x^2}+\dfrac{4}{1+x^4}.$

Here it is simpler first to combine the first and second fractions, next to combine this result and the third fraction, then this last result and the fourth fraction, as below:

$$\frac{1}{1-x}+\frac{1}{1+x}\equiv\frac{1+x+(1-x)}{1-x^2}\equiv\frac{2}{1-x^2};$$

$$\frac{2}{1-x^2}+\frac{2}{1+x^2}\equiv\frac{2(1+x^2)+2(1-x^2)}{1-x^4}\equiv\frac{4}{1-x^4};$$

$$\frac{4}{1-x^4}+\frac{4}{1+x^4}\equiv\frac{4(1+x^4)+4(1-x^4)}{1-x^8}\equiv\frac{8}{1-x^8}.$$

43. $\dfrac{2}{x-2}-\dfrac{1}{x+2}-\dfrac{x+6}{x^2+4}.$

44. $\dfrac{a}{a-x}+\dfrac{a}{a+x}+\dfrac{2a^2}{a^2+x^2}.$

45. $\dfrac{3-x}{1-3x}-\dfrac{3+x}{1+3x}-\dfrac{1-16x}{9x^2-1}.$

46. $\dfrac{1}{x-1}-\dfrac{1}{2(x+1)}-\dfrac{x+3}{2(x^2+1)}.$

47. $\dfrac{a}{a-x}+\dfrac{a}{a+x}+\dfrac{2a^2}{a^2+x^2}+\dfrac{4a^4}{a^4+x^4}.$

48. $\dfrac{1}{x-3}-\dfrac{3}{x-1}+\dfrac{3}{x+1}-\dfrac{1}{x+3}.$

49. $\dfrac{1}{x-2}-\dfrac{4}{x-1}+\dfrac{6}{x}-\dfrac{4}{x+1}+\dfrac{1}{x+2}.$

50. $\dfrac{2}{x^2-3x+2}+\dfrac{2}{x^2-x-2}-\dfrac{1}{x^2-1}.$

The expression

$$\equiv\frac{2}{(x-2)(x-1)}+\frac{2}{(x-2)(x+1)}-\frac{1}{(x-1)(x+1)}$$

$$\equiv\frac{2(x+1)+2(x-1)-(x-2)}{(x-2)(x-1)(x+1)}$$

$$\equiv\frac{3x+2}{(x-2)(x-1)(x+1)}.$$

51. $\dfrac{1}{x^2-9x+20}+\dfrac{1}{x^2-11x+30}.$

52. $\dfrac{1}{x^2-7x+12}-\dfrac{1}{x^2-5x+6}.$

53. $\dfrac{1}{2x^2-x-1}-\dfrac{1}{2x^2+x-3}.$

54. $\dfrac{1}{2x^2-x-1}-\dfrac{3}{6x^2-x-2}.$

55. $\dfrac{4}{4-7a-2a^2}-\dfrac{3}{3-a-10a^2}.$

56. $\dfrac{5}{5+x-18x^2}-\dfrac{2}{2+5x+2x^2}.$

57. $\dfrac{5x}{2(x+1)(x-3)}-\dfrac{15(x-1)}{16(x-3)(x-2)}-\dfrac{9(x+3)}{16(x+1)(x-2)}.$

58. $\dfrac{x}{x^2+5x+6}+\dfrac{15}{x^2+9x+14}-\dfrac{12}{x^2+10x+21}.$

59. $\dfrac{3}{x^2-1}+\dfrac{4}{2x+1}+\dfrac{4x+2}{2x^2+3x+1}.$

60. $\dfrac{24x}{9-12x+4x^2}-\dfrac{3+2x}{3-2x}+\dfrac{3-2x}{3+2x}.$

61. $\dfrac{1}{x^2+5ax+6a^2}-\dfrac{2}{x^2+4ax+3a^2}+\dfrac{1}{x^2+3ax+2a^2}.$

62. $\dfrac{2a}{(x-2a)^2}-\dfrac{x-a}{x^2-5ax+6a^2}+\dfrac{2}{x-3a}.$

63. $\dfrac{1}{a}-\dfrac{4}{a+1}+\dfrac{6}{a+2}-\dfrac{4}{a+3}+\dfrac{1}{a+4}.$

64. $\dfrac{1}{x^2-5x+6}-\dfrac{2}{x^2-4x+3}+\dfrac{1}{x^2-3x+2}.$

65. $\dfrac{1}{8-8x}-\dfrac{1}{8+8x}+\dfrac{x}{4+4x^2}-\dfrac{x}{2+2x^4}.$

66. $\dfrac{1}{6a-18}-\dfrac{1}{6a+18}-\dfrac{1}{a^2+9}+\dfrac{18}{a^4+81}.$

67. $\dfrac{1}{(a-b)(a-c)}+\dfrac{1}{(b-c)(b-a)}+\dfrac{1}{(c-a)(c-b)}.$

In examples of this kind it is best for beginners to arrange all the factors in the denominators of the fractions so that a precedes b or c, and b precedes c.

We therefore change $b-a$ into $-(a-b)$, $c-a$ into $-(a-c)$, and $c-b$ into $-(b-c)$. The expression then becomes

$$\frac{1}{(a-b)(a-c)}-\frac{1}{(a-b)(b-c)}+\frac{1}{(a-c)(b-c)}.$$

The L. C. M. of the denominators is $(a-b)(a-c)(b-c)$;

$$\therefore \text{ the expression} \equiv \frac{b-c-(a-c)+(a-b)}{(a-b)(a-c)(b-c)} \equiv 0.$$

68. $\dfrac{c}{(b-c)(b-a)} + \dfrac{a}{(c-a)(c-b)} + \dfrac{b}{(a-b)(a-c)}$.

69. $\dfrac{z}{(y-x)(z-x)} + \dfrac{x}{(y-z)(y-x)} + \dfrac{y}{(z-x)(z-y)}$.

70. $\dfrac{y+z}{(y-x)(z-x)} + \dfrac{z+x}{(y-z)(y-x)} + \dfrac{x+y}{(z-x)(z-y)}$.

MULTIPLICATION AND DIVISION OF FRACTIONS.

180. Product of fractions. See § 91.

Ex. 1. $\dfrac{x+2}{x+3} \times \dfrac{x+3}{x+4} \times \dfrac{x+4}{x+2} \equiv \dfrac{(x+2)(x+3)(x+4)}{(x+3)(x+4)(x+2)} \equiv 1.$ § 91

Ex. 2. Simplify $\dfrac{x^2-xy}{3y^2} \times \dfrac{xy+y^2}{x(x+2y)} \times \dfrac{2xy}{x^2-y^2}$.

The factors common to numerator and denominator can be cancelled before the multiplication is performed, as below:

$$\text{The expression } \equiv \frac{\cancel{x}\cancel{(x-y)}}{3\cancel{y^2}} \times \frac{\cancel{y}\cancel{(x+y)}}{\cancel{x}(x+2y)} \times \frac{2x\cancel{y}}{\cancel{(x-y)}\cancel{(x+y)}}$$

$$\equiv \frac{2x}{3(x+2y)}.$$

181. To multiply a fraction by any number,

Multiply the numerator, or divide the denominator, by that number.

Proof. $\dfrac{a}{b} \times m \equiv \dfrac{a}{b} \times \dfrac{m}{1} \equiv \dfrac{am}{b}$ § 91

$\equiv a/(b \div m).$ § 173

Ex. $\dfrac{5a^2}{8b^2} \times 4 \equiv \dfrac{5a^2 \times 4}{8b^2}$ or $\dfrac{5a^2}{8b^2 \div 4}$.

182. *The reciprocal of a fraction is equal to the fraction inverted.*

That is, $\mathbf{1 \div (a/b) \equiv b/a}$.

Proof. b/a multiplied by the divisor a/b is equal to the dividend 1; hence b/a is the quotient.

183. To divide by a fraction,

Multiply by the reciprocal of the fraction.

Proof. Dividing by a number gives the same result as multiplying by its reciprocal (§ 87).

Ex. 1. $\frac{3a}{5b} \div \frac{2x}{7y} \equiv \frac{3a}{5b} \times \frac{7y}{2x} \equiv \frac{21ay}{10bx}.$

Ex. 2.
$$\frac{x-a}{x^3+a^3} \div \frac{x^3-a^3}{x+a} \equiv \frac{x-a}{x^3+a^3} \times \frac{x+a}{x^3-a^3}$$
$$\equiv \frac{1}{(x^2-ax+a^2)(x^2+ax+a^2)}$$
$$\equiv \frac{1}{x^4+a^2x^2+a^4}.$$

184. To divide a fraction by any number,

Divide the numerator, or multiply the denominator, by that number.

Proof. $$\frac{a}{b} \div m \equiv \frac{a}{b} \times \frac{1}{m} \equiv \frac{a}{bm} \qquad \text{§ 91}$$
$$\equiv (a \div m)/b. \qquad \text{§ 173}$$

Exercise 73.

Simplify each of the following expressions:

1. $\frac{2a}{3c} \times \frac{3c}{4a}.$

2. $\frac{2a}{3b} \times \frac{6c}{5a} \times \frac{5x}{2y}.$

3. $\frac{2a}{5b} \times \frac{5c}{x} \times \frac{x^2b}{y}.$

4. $\frac{2a^2}{bc} \div \frac{3abx}{c^2y}.$

5. $\frac{3axy}{5b^2} \div \frac{6a^2y^2}{10bx^2}.$

6. $\frac{b}{a} \times \frac{x}{c} \div \frac{c}{a}.$

7. $\frac{a^2}{b^2} \times \frac{x^2}{y^2} \div \frac{a^2}{y^2}.$

8. $\frac{3a^2}{4b} \times \frac{2c}{7ax} \div \frac{3a}{7bx^2}.$

9. $\frac{5y^2}{7a^3} \times \frac{21c_3}{4ax} \div \frac{35c^2y}{7a^3x}.$

10. $\frac{2b}{3a} \div \frac{x}{y} \times \frac{9a^2}{4b^2}.$

11. $\dfrac{x-y}{x^2+xy} \times \dfrac{x+y}{xy-y^2}.$

12. $\dfrac{x^2+2x}{x^2-9} \times \dfrac{x^2-3x}{x^2-4}.$

13. $\dfrac{x^2-y^2}{x^2-4y^2} \times \dfrac{x-2y}{x+y}.$

14. $\dfrac{a+b}{a^3-a^2b} \times \dfrac{ab-b^2}{ab+a^2}.$

15. $\dfrac{x^3+3x^2}{x+4} \div \dfrac{x+3}{x^2+4x}.$

16. $\dfrac{a+4b}{a^2+5ab} \div \dfrac{ab+4b^2}{a^3+5a^2b}.$

17. $\dfrac{x-1}{x-2} \times \dfrac{x-2}{x-3} \div \dfrac{x-4}{x-3}.$

18. $\dfrac{x^3-a^3}{x^2-4a^2} \times \dfrac{x+2a}{x-a}.$

19. $\dfrac{a^3-x^3}{a^3+x^3} \div \dfrac{(a-x)^2}{a^2-x^2}.$

20. $\dfrac{14x^2-7x}{12x^3+24x^2} \div \dfrac{2x-1}{x^2+2x}.$

21. $\dfrac{16x^2-9a^2}{x^2-4} \div \dfrac{4x-3a}{x-2}.$

22. $\dfrac{a^2b^2+3ab}{4a^2-1} \div \dfrac{ab+3}{2a+1}.$

23. $\dfrac{x^2-14x-15}{x^2-4x-15} \div \dfrac{x^2-12x-45}{x^2-6x-27}.$

24. $\dfrac{x^3-6x^2+36x}{x^2-49} \div \dfrac{x^4+216x}{x^2-x-42}.$

25. $\dfrac{x^2-x-20}{x^2-25} \div \dfrac{x+1}{x^2+5x} \div \dfrac{x^2+2x-8}{x^2-x-2}.$

26. $\dfrac{x^2-18x+80}{x^2-5x-50} \div \dfrac{x^2-15x+56}{x^2-6x-7} \times \dfrac{x+5}{x-1}.$

27. $\dfrac{x^2-8x-9}{x^2-17x+72} \times \dfrac{x^2-25}{x^2-1} \div \dfrac{x^2+4x-5}{x^2-9x+8}.$

28. $\dfrac{x^4-8x}{x^2-4x-5} \times \dfrac{x^2+2x+1}{x^3-x^2-2x} \div \dfrac{x^2+2x+4}{x-5}.$

29. $\dfrac{(a+b)^2}{(a-b)^3} \div \dfrac{a^4-b^4}{(a^2-b^2)^2} \div \dfrac{(a+b)^3}{a^2+b^2}.$

30. $\dfrac{(a-b)^2-c^2}{(a-c)^2-b^2} \div \dfrac{c^2-(a-b)^2}{b^2-(c-a)^2}.$

31. $\dfrac{x^6+y^6}{x^6-y^6}\times\dfrac{x-y}{x+y}\div\dfrac{x^4-x^2y^2+y^4}{x^4+x^2y^2+y^4}.$

32. $\left(1-\dfrac{2\,xy}{x^2+y^2}\right)\div\left(\dfrac{x^3-y^3}{x-y}-3\,xy\right).$

First reduce each of the mixed expressions to fractions.

33. $\left(a+\dfrac{ab}{a-b}\right)\left(b-\dfrac{ab}{a+b}\right).$

34. $\dfrac{x^2+xy}{x^2+y^2}\times\left(\dfrac{x}{x-y}-\dfrac{y}{x+y}\right).$

35. $\left(\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}\right)\div\left(\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}\right).$

36. $\left(a+\dfrac{3\,x^2}{a}\right)\times\left(\dfrac{a^2}{3\,x^2}-1\right)\div\dfrac{a}{x}.$

37. $\dfrac{a^3+b^3+3\,ab(a+b)}{2\,a(a-b)^2}\times b^2\left(a-\dfrac{b^2}{a}\right)\div\left(1+\dfrac{b}{a}\right).$

38. $\dfrac{4\,x^2+x-14}{6\,xy-14\,y}\times\dfrac{4\,x^2}{x^2-4}\times\dfrac{x-2}{4\,x-7}\div\dfrac{2\,x^2+4\,x}{3\,x^2-x-14}.$

39. $\dfrac{x^2+x-2}{x^2-x-20}\times\dfrac{x^2+5\,x+4}{x^2-x}\div\left(\dfrac{x^2+3\,x+2}{x^2-2\,x-15}\times\dfrac{x+3}{x^2}\right).$

40. $\dfrac{4\,x^2-16\,x+15}{2\,x^2+3\,x+1}\times\dfrac{x^2-6\,x-7}{2\,x^2-17\,x+21}\times\dfrac{4\,x^2-1}{4\,x^2-20\,x+25}.$

41. $\dfrac{(a+b)^2-c^2}{a^2+ab-ac}\times\dfrac{a}{(a+c)^2-b^2}\times\dfrac{(a-b)^2-c^2}{ab-b^2-bc}.$

42. $\dfrac{a^2+2\,ab+b^2-c^2}{a^2-b^2-c^2-2\,bc}\times\dfrac{a^2-2\,ac+c^2-b^2}{b^2-2\,bc+c^2-a^2}.$

43. $\dfrac{x^2-64}{x^2+24\,x+128}\times\dfrac{x^2+12\,x-64}{x^3-64}\div\dfrac{x^2-16\,x+64}{x^2+4\,x+16}.$

185. A **complex fraction** is a fraction whose numerator and denominator, either or both, are *fractional* expressions.

E.g., $\dfrac{\frac{a+b}{c-d}}{\frac{a}{x}+\frac{c}{y}}$, or $\dfrac{a+b}{c-d}\Big/\left(\dfrac{a}{x}+\dfrac{c}{y}\right)$, is a complex fraction.

Observe that a *heavy* line is drawn between the numerator and denominator of the complex fraction.

Ex. 1. $$\frac{a}{b}\Big/\frac{x}{y}\equiv\frac{a}{b}\times\frac{y}{x}\equiv\frac{ay}{bx}.$$

Ex. 2. $$\left(\frac{x}{y}+1\right)\Big/\left(\frac{y}{x}+1\right)\equiv\frac{x+y}{y}\Big/\frac{y+x}{x}$$
$$\equiv\frac{x+y}{y}\times\frac{x}{y+x}=\frac{x}{y}.$$

Sometimes the easiest way to simplify a complex fraction is to multiply its numerator and denominator by the L. C. M. of the denominators of their fractional terms.

Ex. 3. $$\frac{\frac{a+x}{a-x}-\frac{a-x}{a+x}}{\frac{a+x}{a-x}+\frac{a-x}{a+x}}\equiv\frac{\left(\frac{a+x}{a-x}-\frac{a-x}{a+x}\right)(a-x)(a+x)}{\left(\frac{a+x}{a-x}+\frac{a-x}{a+x}\right)(a-x)(a+x)}$$
$$\equiv\frac{(a+x)^2-(a-x)^2}{(a+x)^2+(a-x)^2}$$
$$\equiv\frac{2\,ax}{a^2+x^2}.$$

Here $(a-x)(a+x)$ is the L. C. M. of the denominators of the fractional terms in the numerator and denominator of the complex fraction.

Ex. 4. $$\frac{x+\frac{a^2}{x}}{x-\frac{a^4}{x^3}}\equiv\frac{\left(x+\frac{a^2}{x}\right)x^3}{\left(x-\frac{a^4}{x^3}\right)x^3}\equiv\frac{x^4+a^2x^2}{x^4-a^4}\equiv\frac{x^2}{x^2-a^2}.$$

Ex. 5. Simplify $$\frac{x}{x-\cfrac{x+2}{x+2-\cfrac{x+1}{x}}}.$$

In a fraction of this kind, called a *continued fraction*, we first simplify the lowest complex fraction as below:

$$\frac{x}{x-\dfrac{x+2}{x+2-\dfrac{x+1}{x}}} \equiv \frac{x}{x-\dfrac{(x+2)x}{(x+2)x-(x+1)}} \qquad \S\ 83$$

$$\equiv \frac{x}{x-\dfrac{x^2+2x}{x^2+x-1}}$$

$$\equiv \frac{x(x^2+x-1)}{x(x^2+x-1)-(x^2+2x)}$$

$$\equiv \frac{x^2+x-1}{x^2-3}.$$

Exercise 74.

Simplify each of the following fractional expressions:

1. $\dfrac{\dfrac{m}{n}-\dfrac{l}{m}}{\dfrac{a}{m}-\dfrac{b}{n}}.$

2. $\dfrac{a+\dfrac{b}{d}}{x-\dfrac{y}{d}}.$

3. $\dfrac{2+\dfrac{3a}{4b}}{a+\dfrac{8b}{3}}.$

4. $\dfrac{3a+\dfrac{7b}{8c}}{3c+\dfrac{7b}{8a}}.$

5. $\dfrac{x}{x-\dfrac{m}{n}}.$

6. $\dfrac{1-\dfrac{y^2}{x^2}}{1+\dfrac{y^2}{x^2}}.$

7. $\dfrac{\dfrac{a}{b}+\dfrac{c}{d}}{\dfrac{m}{n}+\dfrac{k}{p}}.$

8. $\dfrac{x-\dfrac{1}{x}}{1+\dfrac{1}{x}}.$

9. $\dfrac{x+5+\dfrac{6}{x}}{1+\dfrac{6}{x}+\dfrac{8}{x^2}}.$

10. $\dfrac{\dfrac{1}{x}-\dfrac{2}{x^2}-\dfrac{3}{x^3}}{\dfrac{9}{x}-x}.$

11. $\dfrac{2x^2-x-6}{\dfrac{4}{x^2}-1}.$

12. $\dfrac{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}{1-\dfrac{a^2+b^2}{(a+b)^2}}.$

13. $\dfrac{\dfrac{a}{x^2}+\dfrac{x}{a^2}}{\dfrac{1}{a^2}-\dfrac{1}{ax}+\dfrac{1}{x^2}}.$

14. $1+\dfrac{x}{1+x\dfrac{2x^2}{1-x}}.$

15. $\dfrac{1}{a-\dfrac{a^2-1}{a+\dfrac{1}{a-1}}}.$

16. $\dfrac{a}{x+\dfrac{m}{y+\dfrac{n}{z}}}.$

17. $\dfrac{1}{1-\dfrac{1+x}{x-\dfrac{1}{x}}}.$

18. $\dfrac{x-2}{x-2-\dfrac{x}{x-\dfrac{x-1}{x-2}}}.$

19. $\dfrac{1}{x-\dfrac{1}{x+\dfrac{1}{x}}}-\dfrac{1}{x+\dfrac{1}{x-\dfrac{1}{x}}}.$

20. $\left\{x-y-\dfrac{1}{x-y+\dfrac{xy}{x-y}}\right\}\times\dfrac{x^3+y^3}{x^2-y^2}.$

21. $\left\{x+y-\dfrac{1}{x+y-\dfrac{xy}{x+y}}\right\}\times\dfrac{x^3-y^3}{x^2-y^2}.$

22. $\left\{1-\dfrac{1-x}{1+x}+\dfrac{1+2x^2}{1-x^2}\right\}\times\dfrac{x+1}{2x+1}.$

23. $\dfrac{x-2-\dfrac{1}{x-2}}{x-2-\dfrac{4}{x-5}}\times\dfrac{x-4-\dfrac{4}{x-4}}{x-4-\dfrac{1}{x-4}}.$

24. $\left(\dfrac{a^2+b^2}{2ab}-1\right)\dfrac{ab^2}{a^3+b^3}\div\dfrac{4ab(a+b)}{a^2-ab+b^2}.$

25. $\dfrac{\left\{1+\dfrac{c}{a+b}+\dfrac{c^2}{(a+b)^2}\right\}\left\{1-\dfrac{c^2}{(a+b)^2}\right\}}{\left\{1-\dfrac{c^3}{(a+b)^3}\right\}\left\{1+\dfrac{c}{a+b}\right\}}.$

26. $\left(\dfrac{x^2}{y}-\dfrac{y^2}{x}\right)\left(\dfrac{x}{y^2}+\dfrac{y}{x^2}\right)\div\left(\dfrac{x^2}{y}+\dfrac{y^2}{x}\right)\left(\dfrac{x}{y^2}-\dfrac{y}{x^2}\right).$

27. $\dfrac{a^6+b^6}{a^6-b^6}\times\dfrac{a-b}{a^2-b^2}\div\dfrac{a^4-a^2b^2+b^4}{a^4+a^2b^2+b^4}$.

28. $\left(\dfrac{x+2}{2x+3}-\dfrac{4x+5}{5x+6}\right)\div\left(\dfrac{2x+3}{3x+4}-\dfrac{3x+4}{4x+5}\right)$.

29. $\left(\dfrac{1+x}{1+x^2}-\dfrac{1+x^2}{1+x^3}\right)\div\left(\dfrac{1+x^2}{1+x^3}-\dfrac{1+x^3}{1+x^4}\right)$.

30. $\left\{\left(\dfrac{a-b}{a+b}\right)^2-2\dfrac{a-b}{a+b}+1\right\}\div\left\{\left(\dfrac{a+b}{a-b}\right)^2-2\dfrac{a+b}{a-b}+1\right\}$.

186. Power of a fraction. *The **n**th power of a fraction is equal to the **n**th power of its numerator divided by the **n**th power of its denominator; and conversely.*

That is, $$(a/b)^n \equiv a^n/b^n.$$

Proof. $\left(\dfrac{a}{b}\right)^n \equiv \dfrac{a}{b}\cdot\dfrac{a}{b}\cdot\dfrac{a}{b}\cdots$ to n factors by notation

$\equiv \dfrac{aaa\cdots \text{to } n \text{ factors}}{bbb\cdots \text{to } n \text{ factors}}$ § 91

$\equiv a^n/b^n.$ by notation

Ex. 1. $\left(-\dfrac{2x^2y^3}{3a^3b^2c}\right)^3 \equiv (-1)^3\dfrac{(2x^2y^3)^3}{(3a^3b^2c)^3}$ §§ 119, 186

$\equiv -\dfrac{8x^6y^9}{27a^9b^6c^3}.$ §§ 118, 119

Ex. 2. $\dfrac{(x^2-7x+12)^2}{(x-3)^2}\equiv\left[\dfrac{(x-3)(x-4)}{x-3}\right]^2\equiv(x-4)^2.$ § 186

Exercise 75.

Write each of the following powers as a quotient of products:

1. $\left(-\dfrac{2a}{3b^2}\right)^2$.
3. $\left(-\dfrac{2ax}{3by^2}\right)^3$.
5. $\left(\dfrac{2a^3x^2}{3b^2y^3}\right)^2$.
2. $\left(-\dfrac{3ax^3}{4by^2}\right)^2$.
4. $\left(-\dfrac{c^2x^3}{2by^2}\right)^3$.
6. $\left(-\dfrac{a^2xy^2}{2bc^2z^3}\right)^3$.

7. $\left(-\frac{ax^2b}{cy^2}\right)^4$.

10. $\left(-\frac{ax^2y}{bc^2z}\right)^7$.

13. $\left(-\frac{x^2y}{ab^2}\right)^{2n}$.

8. $\left(\frac{2\,axy^2}{b^2cz}\right)^5$.

11. $\left(-\frac{2\,xyz^2}{3\,a^2bc^3}\right)^4$.

14. $\left(-\frac{xy}{ab}\right)^{2n+1}$.

9. $\left(-\frac{abc^2}{x^2yz^3}\right)^6$.

12. $\left(-\frac{2\,ax^2y}{b^2c^3z}\right)^5$.

15. $\left(-\frac{x^2y^c}{a^cb^m}\right)^{2n}$.

Simplify each of the following expressions:

16. $\frac{(a^2-1)^3}{(a-1)^3} \equiv \left(\frac{a^2-1}{a-1}\right)^3 \equiv (a+1)^3.$

17. $\frac{(x^2-a^2)^5}{(x-a)^5}$.

18. $\frac{(x^2+xy)^6}{(x+y)^6}$.

19. $\frac{(a^4-b^4)^3}{(a^2-b^2)^3}$.

20. $\frac{(a^3-1)^4}{(a^2+a+1)^4}$.

22. $\left(\frac{x+1}{y+1}\right)^2 \times \frac{y^2-1}{x^3+1}$.

21. $\frac{(x+5)^5}{(x^2+3\,x-10)^5}$.

23. $\left(\frac{a^2}{b}+\frac{x}{y^2}\right)\left(\frac{a^2}{b}-\frac{x}{y^2}\right)$.

24. $\left(\frac{x^2y}{2\,ab^2}+\frac{c^2n}{m^2z}\right)\left(\frac{x^2y}{2\,ab^2}-\frac{c^2n}{m^2z}\right)$.

Expand each of the following powers:

25. $\left(\frac{1}{x}+\frac{1}{y}\right)^2$.

28. $\left(\frac{x^2}{a^2}-\frac{a^2}{x^2}\right)^2$.

31. $\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2$.

26. $\left(\frac{1}{a^2}-\frac{1}{b^2}\right)^3$.

29. $\left(\frac{a^2}{b^2}-\frac{x^2}{y^2}\right)^3$.

32. $\left(\frac{a}{x}+\frac{b}{y}-\frac{c}{z}\right)^2$.

27. $\left(\frac{a}{b}-\frac{x}{y}\right)^4$.

30. $\left(\frac{a}{b}+\frac{b}{a}\right)^5$.

33. $\left(\frac{a}{x}-\frac{x}{a}-\frac{b}{y}\right)^2$.

Factor each of the following expressions:

34. $4-\frac{4\,x}{y}+\frac{x^2}{y^2}$.

36. $\frac{x^2}{y^2}-\frac{2\,ax}{by}+\frac{a^2}{b^2}$.

35. $\frac{x^2}{y^2}+\frac{10\,x}{y}+25$.

37. $\frac{x^2}{4\,y^2}-\frac{2\,x}{y}+4$.

38. $\dfrac{64\,x^2}{9\,y^2}+\dfrac{32\,x}{3\,y}+4.$

39. $\dfrac{9\,x^2}{25}-2+\dfrac{25}{9\,x^2}.$

40. $\dfrac{x^3}{8}-\dfrac{3\,x^2}{4}+\dfrac{3\,x}{2}-1.$

41. $\dfrac{x^3}{27}+\dfrac{2\,x^2}{3}+4\,x+8.$

42. $8\,x^3-4\,x^2y^2+\frac{2}{3}\,xy^4-\dfrac{y^6}{27}.$

43. $\dfrac{27\,x^3}{64\,y^3}-\dfrac{27\,x^2}{8\,y^2}+\dfrac{9\,x}{y}-8.$

44. $\dfrac{x^2}{y^2}-\dfrac{a^2}{b^2}.$

45. $\dfrac{4\,a^4x^2}{b^2}-\dfrac{9\,y^2}{c^2}.$

46. $\dfrac{x^3}{y^3}-\dfrac{a^3}{b^3}.$

47. $\dfrac{4\,x^2}{c^2}-4+\dfrac{4\,x}{a}-\dfrac{x^2}{a^2}.$

48. $\dfrac{9\,y^2}{a^2}-\dfrac{x^2}{b^2}+\dfrac{2\,ax}{bc}-\dfrac{a^2}{c^2}.$

49. $\dfrac{9\,x^2}{25\,a^2}+\dfrac{6\,bx}{5\,a^2}+\dfrac{b^2}{a^2}-\dfrac{4\,x^2}{c^2}.$

Reduce to its lowest terms each of the following fractions:

50. $\dfrac{x^2-8\,xy+7\,y^2}{x^2-3\,xy-28\,y^2}.$

51. $\dfrac{a^4-b^4}{(a^3-b^3)(a+b)}.$

52. $\dfrac{(a^3-x^3)(a+x)}{(a^3+x^3)(a-x)}.$

53. $\dfrac{(x^6-y^6)(x-y)}{(x^3-y^3)(x^4-y^4)}.$

54. $\dfrac{x^6+x^4-x^2-1}{x^8-x^6+x^2-1}.$

55. $\dfrac{a^5-a^4b-ab^4+b^5}{a^4-a^3b-a^2b^2+ab^3}.$

56. $\dfrac{(x+y+z)^2-(x-y-z)^2}{3\,x(y^2+2\,yz+z^2)}.$

57. $\dfrac{a^4-16}{a^4-4\,a^3+8\,a^2-16\,a+16}.$

CHAPTER XIII

FRACTIONAL EQUATIONS

187. A **fractional equation** is an equation one or both of whose members are fractional with respect to an *unknown*.

E.g., $\frac{2x}{2x-1} - \frac{x-1}{x} = 4$ is a fractional equation in x,

while $\frac{x}{2} + \frac{ax}{b} = \frac{x+2}{a+4}$ is an integral equation in x.

We cannot speak of the *degree* of a fractional equation. The term *degree* as defined in § 101 applies only to an *integral* equation.

188. *If both members of an integral equation are multiplied by the same unknown integral expression M, the derived equation has all the roots of the given equation, and, in addition, those of $M = 0$.*

E.g., if we multiply both members of the equation

$$2x + 1 = x + 3 \tag{1}$$

by $x - 5$; the root 5 is introduced in the derived equation.

Proof. If A and B denote integral expressions in the unknown, and we multiply both members of

$$A = B \tag{1}$$

by any unknown integral expression M, we obtain

$$AM = BM, \text{ or } (A - B)\,M = 0. \tag{2}$$

By § 149, (2) is equivalent to the two equations

$$A - B = 0 \text{ and } M = 0.$$

That is, the roots of $M=0$ are introduced in the derived equation (2) by multiplying both members of (1) by M.

189. *If both members of a fractional equation in* **one** *unknown are multiplied by any integral expression which is* **necessary** *to clear the equation of fractions, the derived integral equation will be equivalent to the given fractional equation.*

Ex. 1. Solve the equation $\dfrac{3}{x-1}=5-x.$ (1)

Multiplying by $x-1$ to clear (1) of fractions, we obtain

$$3=6x-x^2-5.$$

Transpose, $$x^2-6x+8=0.$$

Factor, $$(x-2)(x-4)=0. \qquad (2)$$

No root could be lost, nor could either root of (2) be introduced by multiplying (1) by $x-1$; hence (2) is equivalent to (1).

Therefore, the roots of (1) are 2 and 4.

Ex. 2. Solve $$\frac{3}{x-5}+\frac{2x}{x-3}=5. \qquad (1)$$

Multiplying by $(x-5)(x-3)$ to clear (1) of fractions we obtain

$$3(x-3)+2x(x-5)=5(x-5)(x-3).$$

$$\therefore x^2-11x+28=0.$$

$$\therefore (x-4)(x-7)=0. \qquad (2)$$

No root could be lost nor could either root of (2) be introduced by multiplying by $x-5$ or $x-3$; hence (2) is equivalent to (1).

Therefore, the roots of (1) are 4 and 7.

Proof. By transposing to the first member all the terms of any fractional equation, adding them, and reducing the resulting fraction to its lowest terms, we derive an equation of the form

$$A/B=0, \qquad (1)$$

where A and B have no common factors.

By the preceding principles of equivalent equations, the derived equation (1) is equivalent to the given fractional equation.

We are to prove that (1) is equivalent to the equation

$$A = 0. \tag{2}$$

Any root of (1) reduces A/B to 0. But when A/B is zero, A is zero; hence any root of (1) is a root of (2).

To prove the converse we must first prove that when

$$A = 0, \ B \neq 0.$$

If A and B could become 0 for the same value of x, as a; then by § 132 they would have the factor $x - a$ in common. But by hypothesis A and B have no common factor; hence when $A = 0, \ B \neq 0$.

Hence any root of (2) reduces A to 0 but not B to 0.

Therefore any root of (2) reduces A/B to 0 and is a root of (1).

Hence equations (1) and (2) are equivalent.

Ex. 3. Solve $$1 - \frac{x^2}{x-1} = \frac{1}{1-x} - 6. \tag{1}$$

Transposing and adding the fractions, we have

$$1 - \frac{x^2 - 1}{x - 1} + 6 = 0.$$

$$\therefore 1 - (x + 1) + 6 = 0, \text{ or } x = 6. \tag{2}$$

By §§ 105 and 106, equation (2) is equivalent to (1); hence 6 is the *one* and *only* root of (1).

But if, as would be more natural for the beginner, we should clear equation (1) of fractions by multiplying by $x - 1$, we would obtain

$$x - 1 - x^2 = -1 - 6x + 6.$$

Transpose, $$x^2 - 7x + 6 = 0.$$

$$\therefore (x - 1)(x - 6) = 0, \tag{3}$$

of which the roots are 1 and 6.

As was shown above, to clear equation (1) of fractions it was not *necessary* to multiply by $x-1$; hence multiplying (1) by $x-1$ is the same as multiplying the equivalent integral equation (2) by $x-1$.

In clearing of fractions an equation in one unknown, to avoid introducing roots, the following suggestions should be heeded:

(i) Fractions having a common denominator should be combined.

(ii) Factors common to the numerator and denominator of any fraction should be cancelled.

(iii) When multiplying by a multiple of the denominators, we should always use the L. C. M.

Ex. 4. Solve
$$\frac{x}{x-2}+\frac{x-9}{x-7}=\frac{x+1}{x-1}+\frac{x-8}{x-6}. \qquad (1)$$

Transpose so that each member is a difference,
$$\frac{x}{x-2}-\frac{x+1}{x-1}=\frac{x-8}{x-6}-\frac{x-9}{x-7}.$$

Combine,
$$\frac{2}{(x-2)(x-1)}=\frac{2}{(x-7)(x-6)}; \qquad (2)$$

Clear of fractions, $x^2-13x+42=x^2-3x+2$.
$$\therefore 10x=40, \text{ or } x=4. \qquad (3)$$

Since the root 4 could not be introduced in clearing (2) of fractions, 4 is the root of (1).

Ex. 5. Solve
$$\frac{x-1}{x+1}+\frac{x+5}{x+7}=\frac{x+1}{x+3}+\frac{x+3}{x+5}. \qquad (1)$$

Transpose,
$$\frac{x-1}{x+1}-\frac{x+1}{x+3}=\frac{x+3}{x+5}-\frac{x+5}{x+7}. \qquad (2)$$

Combine,
$$\frac{1}{(x+1)(x+3)}=\frac{1}{(x+5)(x+7)}. \qquad (3)$$

Clear of fractions, $x^2+12x+35=x^2+4x+3$.
$$\therefore 8x=-32, \text{ or } x=-4.$$

Or reducing the improper fractions in (2) to mixed expressions, we have,

$$1-\frac{2}{x+1}-1+\frac{2}{x+3}=1-\frac{2}{x+5}-1+\frac{2}{x+7}.$$

$$\therefore -\frac{1}{x+1}+\frac{1}{x+3}=-\frac{1}{x+5}+\frac{1}{x+7}.$$

Combining these fractions, we obtain equation (3) above.

Since the root -4 could not be introduced in clearing (3) of fractions, -4 is the root of (1).

Exercise 76.

Solve each of the following fractional equations:

1. $\frac{3x-16}{x}=\frac{5}{3}.$

2. $\frac{5x-5}{x+1}=3.$

3. $\frac{x-1}{x+1}=\frac{2}{3}.$

4. $\frac{x-2}{2x-5}=\frac{x-5}{2x-2}.$

5. $\frac{2x-3}{3x-4}=\frac{4x-5}{6x-7}.$

6. $\frac{x}{x+1}=\frac{3x}{x+2}-2.$

7. $\frac{x-1}{x+1}+\frac{1}{x}=1.$

8. $\frac{10-7x}{x-1}=\frac{5}{x+1}-7.$

9. $\frac{1}{4x+6}+\frac{1}{6x+4}=\frac{2}{2x+3}.$

10. $\frac{1}{3x+9}+\frac{2}{5x+1}=\frac{2}{x+3}.$

11. $\frac{3}{2x+3}+\frac{5}{4x+6}=\frac{7}{6x+8}.$

12. $\frac{4x}{x+1}-\frac{x}{x-2}=3.$

13. $\frac{6x}{x-7}-\frac{x}{x-6}=5.$

14. $\frac{2x}{x+3}=\frac{4x}{x+7}-2.$

15. $\frac{1}{x+4}+\frac{2}{x+6}=\frac{3}{x+5}.$

16. $\frac{3}{x+1}-\frac{2}{x+2}=\frac{1}{x+3}.$

17. $\frac{5}{2}\frac{1}{x+4}=\frac{3}{2}\frac{1}{x+2}+\frac{1}{x+6}.$

18. $\frac{3}{2}\frac{1}{x+2}=\frac{5}{4}\frac{1}{x+3}+\frac{1}{4x}.$

19. $\frac{2x-5}{3x-7}=\frac{2x-7}{3x-5}.$

20. $\frac{6x-2}{8x-5}=\frac{3x+7}{4x+8}.$

21. $\dfrac{x+1}{x-1}-\dfrac{x-3}{x+3}=\dfrac{8}{x}.$

22. $\dfrac{x+2}{x-2}-\dfrac{x-2}{x+2}=\dfrac{8}{x+1}.$

23. $\dfrac{x-1}{x+3}=\dfrac{x+2}{x-4}-\dfrac{10}{x}.$

24. $\dfrac{3x}{x+2}=3\dfrac{x-1}{x-2}-\dfrac{9}{x+1}.$

25. $3\dfrac{x-1}{x+1}+2\dfrac{x+1}{x-1}=5.$

26. $5\dfrac{x-2}{x+2}-2\dfrac{x-3}{x+3}=3.$

27. $\dfrac{4}{x^2-1}+\dfrac{1}{x+1}=\dfrac{1}{1-x}.$

28. $\dfrac{3}{x^2-9}+\dfrac{1}{x+3}=\dfrac{2}{3-x}.$

29. $\dfrac{3x+5}{3x-1}+\dfrac{5}{1-9x^2}=\dfrac{8+3x}{1+3x}.$

30. $\dfrac{1}{x+5}-\dfrac{1}{x+6}=\dfrac{1}{x+6}-\dfrac{1}{x+8}.$

31. $\dfrac{1}{x+2}+\dfrac{1}{x+10}=\dfrac{1}{x+4}+\dfrac{1}{x+8}.$

32. $\dfrac{1}{x-5}+\dfrac{1}{x+2}=\dfrac{1}{x-4}+\dfrac{1}{x+1}.$

33. $\dfrac{x}{x-2}+\dfrac{x-9}{x-7}=\dfrac{x+1}{x-1}+\dfrac{x-8}{x-6}.$

34. $\dfrac{x+3}{x+1}+\dfrac{x-6}{x-4}=\dfrac{x+4}{x+2}+\dfrac{x-5}{x-3}.$

35. $\dfrac{x-3}{x-4}-\dfrac{x-4}{x-5}=\dfrac{x-6}{x-7}-\dfrac{x-7}{x-8}.$

36. $\dfrac{x}{x-2}-\dfrac{x+1}{x-1}=\dfrac{x-8}{x-6}-\dfrac{x-9}{x-7}.$

37. $\dfrac{x+5}{x+4}-\dfrac{x-6}{x-7}=\dfrac{x-4}{x-5}-\dfrac{x-15}{x-16}.$

38. $\dfrac{x-7}{x-9}-\dfrac{x-9}{x-11}=\dfrac{x-13}{x-15}-\dfrac{x-15}{x-17}.$

39. $\dfrac{x+3}{x+6}-\dfrac{x+6}{x+9}=\dfrac{x+2}{x+5}-\dfrac{x+5}{x+8}.$

40. $\dfrac{x+2}{x}+\dfrac{x-7}{x-5}-\dfrac{x+3}{x+1}=\dfrac{x-6}{x-4}.$

41. $\dfrac{4x-17}{x-4}+\dfrac{10x-13}{2x-3}=\dfrac{8x-30}{2x-7}+\dfrac{5x-4}{x-1}.$

42. $\dfrac{5x-8}{x-2}+\dfrac{6x-44}{x-7}-\dfrac{10x-8}{x-1}=\dfrac{x-8}{x-6}.$

43. $\dfrac{30+6x}{x+1}+\dfrac{60+8x}{x+3}=14+\dfrac{48}{x+1}.$

44. $\dfrac{25-\frac{1}{3}x}{x+1}+\dfrac{16x+4\frac{1}{5}}{3x+2}=5+\dfrac{23}{x+1}.$

45. $\dfrac{3}{4-2x}+\dfrac{30}{8(1-x)}=\dfrac{3}{2-x}+\dfrac{5}{2-2x}.$

46. $\dfrac{7}{x-4}-\dfrac{60}{5x-30}=\dfrac{10\frac{1}{2}}{3x-12}-\dfrac{8}{x-6}.$

47. $\dfrac{4}{x+3}-\dfrac{2}{x+1}=\dfrac{5}{2x+6}-\dfrac{2\frac{1}{2}}{2x+2}.$

48. $\dfrac{(2x-1)(3x+8)}{6x(x+4)}-1=0.$

In the five following examples first reduce improper fractions to mixed expressions.

49. $\dfrac{5x-64}{x-13}-\dfrac{2x-11}{x-6}=\dfrac{4x-55}{x-14}-\dfrac{x-6}{x-7}.$

50. $\dfrac{x-8}{x-10}+\dfrac{x-4}{x-6}=\dfrac{x-5}{x-7}+\dfrac{x-7}{x-9}.$

51. $\dfrac{x+b}{x-c}+\dfrac{x+c}{x-b}=2.$

52. $\dfrac{x+b}{x-b}=\dfrac{x-c}{x+c}+\dfrac{2(b+c)}{x}.$

53. $\dfrac{mx}{m+x}+\dfrac{nx}{n+x}=m+n.$

54. $\dfrac{x-c}{x-b}=\dfrac{x-b}{x-c}+\dfrac{2(b-c)}{x-b-c}.$

55. $\dfrac{m+r}{x+2n}+\dfrac{n+r}{x+2m}=\dfrac{m+n+2r}{x+m+n}.$

190. Problems which lead to fractional equations.

Prob. 1. The quotient of a certain number increased by 7 divided by the same number diminished by 5 is 4. Find the number.

Let $x =$ the required number.

Then by the conditions of the problem, we have

$$\frac{x+7}{x-5}=4.$$

Whence $x = 9$, the required number.

Prob. 2. The value of a fraction is 1/4. If its numerator is diminished by 2 and its denominator is increased by 2, the resulting fraction will be equal to 1/9. Find the fraction.

Let $x =$ the numerator of the fraction;

then $4x =$ the denominator of the fraction;

and, by the conditions of the problem, we have

$$\frac{x-2}{4x+2}=\frac{1}{9}.$$

Whence $x = 4$, and the required fraction is 4/16.

Exercise 77.

1. The value of a fraction is 1/7. If its numerator is increased by 5 and its denominator by 15, the resulting fraction will be equal to 1/5. Find the fraction.

2. The sum of two numbers is 20, and the quotient of the less divided by the greater is 1/3. Find the numbers.

3. What number added to the numerator and denominator of the fraction 3/7 will give a fraction equal to 2/3?

4. What number must be added to the numerator and subtracted from the denominator of the fraction 5/11, to give its reciprocal?

5. The reciprocal of a number is equal to 7 times the reciprocal of the sum of the number and 5. Find the number.

6. A train ran 240 miles in a certain time. If it had run 6 miles an hour faster, it would have run 48 miles farther in the same time. Find the rate of the train.

7. A number has three digits which increase by 2 from right to left. The quotient of the number divided by the sum of the digits is 48. Find the number.

8. A steamer can run 18 miles an hour in still water. If it can run 96 miles with the current in the same time that it can run 48 miles against the current, what is the rate of the current?

9. A number of men have $ 80 to divide. If $ 150 were divided among 2 more men, each one would receive $ 5 more. Find the number of men.

10. The circumference of the hind wheel of a wagon exceeds the circumference of the front wheel by 4 feet. In running 200 yards the front wheel makes 10 more revolutions than the hind wheel. What is the circumference of each wheel?

11. A number has two digits which increase by 4 from right to left. If the digits are interchanged and the resulting number is divided by the first number the quotient will be 4/7. Find the number.

12. A train runs 10 miles farther in an hour than a man rides on a bicycle in the same time. If it takes the man 5 hours longer to ride 352 miles than it takes the train to run the same distance, what is the rate of the train?

13. A tank can be filled with one pipe in 30 minutes, by a second pipe in 40 minutes, by a third in 50 minutes. How long will it take to fill it with them all running together?

14. A can do a piece of work in $3\frac{1}{4}$ days, B in $2\frac{1}{2}$ days, C in $3\frac{1}{3}$ days. If A, B, and C work together, how long will it take to do the work?

15. A cistern can be filled in 15 minutes by two pipes, A and B, running together; after A has been running by itself for 5 minutes, B is also turned on, and the cistern is filled in 13 minutes more. In what time would it be filled by each pipe separately?

16. A man, woman, and child could reap a field in 30 hours, the man doing half as much again as the woman, and the woman two-thirds as much again as the child. How many hours would they each take to do it separately?

17. A and B ride 100 miles from P to Q. They ride together at a uniform rate until they are within 30 miles of Q, when A increases his rate by 1/5 of his previous rate. When B is within 20 miles of Q he increases his rate by 1/2 of his previous rate, and arrives at Q 10 minutes earlier than A. At what rate did A and B first ride?

18. A and B can reap a field together in 12 hours, A and C in 16 hours, and A by himself in 20 hours. In what time could B and C together reap it? In what time could A, B, and C together reap it?

19. The sum of two numbers is n, and the quotient of the less divided by the greater is a/b. Find the numbers.

20. The reciprocal of a number is n times the reciprocal of the sum of the number and a. Find the number.

21. A train ran a miles in a certain time. If it had run b miles an hour faster, it would have run c miles further in the same time. Find the rate of the train.

22. A steamer can run a miles an hour in still water. If it can run b miles with the current in the same time that it can run c miles against the current, what is the rate of the current?

23. The value of a fraction is $1/a$. If its numerator is increased by m and its denominator by n, the resulting fraction will be equal to $1/b$. Find the fraction.

CHAPTER XIV

SYSTEMS OF LINEAR EQUATIONS

191. Equations in two or more unknowns. In the equation

$$y = 3x + 2, \tag{1}$$

where x and y are both unknowns, y has one and only one value for each value of x.

E.g., when $x = 1$, $y = 5$; when $x = 2$, $y = 8$; when $x = 3$, $y = 11$; when $x = 4$, $y = 14$, etc.

That is, equation (1) restricts x and y to *sets of values.*

In like manner, *any equation in two or more unknowns* restricts its unknowns to *sets of values.*

192. A **solution** of an equation in two or more unknowns is any *set* of values of the unknowns which renders the equation an identity.

E.g., if in the equation

$$y = 3x + 2 \tag{1}$$

we put 3 for x and 11 for y, we obtain the identity

$$11 = 3 \times 3 + 2.$$

Hence 3 and 11, as a set of values of x and y, is one solution of (1); 2 and 8 is another solution; and so on.

NOTE. The word *solution* denotes either the *process* of solving or the *result* obtained by solving. The word is here used in the latter sense.

A *root* of an equation in one unknown is often called a *solution.*

193. The **degree** of an integral equation in two or more unknowns is the degree of that term which is of the highest degree in the unknowns.

E.g., $ax + by = 7$ is a linear equation in x and y; while $ax^2 + by = c$ or $cxy + 3x = 2$ is a quadratic equation in x and y.

194. Two equations are said to be **equivalent** when every solution of each equation is a solution of the other.

195. The following **principles concerning the equivalence of equations,** which have been proved for equations in one unknown, hold true for all equations:

(i) *If for any expression in an equation an identical expression is substituted, the derived equation will be equivalent to the given one* (§ 105).

(ii) *If identical expressions are added to or subtracted from both members of an equation, the derived equation will be equivalent to the given one* (§ 106).

(iii) *If both members of an equation are multiplied or divided by the same known expression, not denoting zero, the derived equation will be equivalent to the given one* (§§ 108, 110).

(iv) *If one member of an equation is zero, and the other member is the product of two or more integral factors, the equations formed by putting each of these factors equal to zero are together equivalent to the given equation* (§ 149).

E.g., the equation

$$(x + 2y - 4)(2x - 3y + 1) = 0$$

is equivalent to the two equations

$$x + 2y - 4 = 0 \text{ and } 2x - 3y + 1 = 0.$$

(v) *If both members of an integral equation are multiplied by the same unknown integral expression M, the derived equation has all the solutions of the given equation, and in addition those of $M = 0$* (§ 188).

Proof. If in the proofs of these principles for equations in one unknown we substitute the word "solution" for the word "root," the proofs will apply to equations in any number of unknowns.

Exercise 78.

Of the following equations state which are equivalent to the equation $2x + y = 3$, and give the reason:

1. $(4x + 2y)/2 = 3$.
2. $3x + y = x + 3$.
3. $x + y = 3 - x$.
4. $6x + 3y = 9$.
5. $(2x + y)/3 = 1$.
6. $4x + 3y = 6 + y$.

State to what two linear equations each of the following quadratic equations is equivalent, and give the reason:

7. $(x-y)(x+2y+1)=0$.
8. $(y-x)x+2y(x-y)=0$.

Obtain ten solutions of each of the following equations:

9. $2x + y = 3$.
10. $2x + 3y = 6$.
11. $2x - 3y = 4$.

12. How many solutions has a single equation in two unknowns?

13. By (iii) in § 195, show that the two equations

$$ax + by = c \text{ and } a'x + b'y = c'$$

are equivalent when $a'/a = b'/b = c'/c$.

196. Independent equations.

Prob. If the sum of two numbers is 10 and their difference is 4, what are the two numbers?

Let $x =$ the less number

and $y =$ the greater number.

Then by the *first* condition we have the equation

$$y + x = 10; \qquad (1)$$

and by the *second* condition we have the equation

$$y - x = 4. \qquad (2)$$

In (1), when $x=1$, $y=9$; when $x=2$, $y=8$; when $x=3$, $y=7$, etc.

In (2), when $x=1$, $y=5$; when $x=2$, $y=6$; when $x=3$, $y=7$, etc.

Hence, 3 and 7 is a set of values of x and y which will satisfy each of the two different conditions expressed by equations (1) and (2), and are therefore the required numbers.

Equations, like (1) and (2), which express different conditions are called **independent** equations.

Observe that *independent* equations express *different* relations between their unknowns, while *equivalent* equations express the *same* relation.

Any solution as $x=3$, $y=7$, can be written briefly 3, 7, it being understood that the value of x is written first.

197. Systems of equations. Two or more equations are said to be **simultaneous,** when the unknowns are restricted to the set or sets of values which satisfy all the equations.

A group of two or more simultaneous equations is called a **system of equations.**

E.g., equations (1) and (2) in § 196 are *simultaneous*, and form a *system of equations.*

198. A **solution** of a system of equations is a set of values of its unknowns which satisfies all its equations.

E.g., 3, 7 is a solution of the system of equations, (1) and (2), in § 196.

To **solve** a system of equations is to find all its solutions.

199. Equivalent systems. Two systems of equations are said to be *equivalent* when every solution of each system is a solution of the other system.

E.g., the systems (a) and (b),

$$\left.\begin{aligned} x+2y&=5, &(1)\\ 4x-y&=2, &(2)\end{aligned}\right\}(a)\qquad \left.\begin{aligned} x&=5-2y, &(3)\\ 4(5-2y)-y&=2, &(4)\end{aligned}\right\}(b)$$

are equivalent; for each system has the solution 1, 2; and, as will be proved later, neither system has any other solution.

Observe that (3) is obtained by solving (1) for x, and (4) by putting in (2) the value of x given in (3); x therefore does not appear in (4), and is said to have been *eliminated.*

200. Elimination is the process of deriving from two or more equations a new equation involving one less unknown than the equations from which it is derived.

The unknown which does not appear in the derived equation is said to have been *eliminated;* as x in § 199.

There are in common use two methods of elimination:

I. *Elimination by substitution or comparison.*

II. *Elimination by addition or subtraction.*

201. In this chapter we shall use three principles concerning the *equivalence of systems* of equations. For convenience of reference we shall number them, (i), (ii), (iii).

(i) **Equivalent equations.** *If any equation of a system is replaced by an equivalent equation, the derived system will be equivalent to the given system.*

E.g., since equation (3) is equivalent to (1), and (4) to (2), system (b) is equivalent to system (a).

$$\left.\begin{array}{lr} 3x+2y=8, & (1) \\ 4x-3y=5, & (2) \end{array}\right\}(a) \qquad \left.\begin{array}{lr} 9x+6y=24, & (3) \\ 8x-6y=10. & (4) \end{array}\right\}(b)$$

The only solution of either system is 2, 1.

Ex. Solve the system

$$\left.\begin{array}{lr} 3+4x=15, & (1) \\ 2+3y=8. & (2) \end{array}\right\}(a)$$

From (1),

$$\left.\begin{array}{lr} x=3. & (3) \\ y=2. & (4) \end{array}\right\}(b)$$

From (2),

Since equation (3) is equivalent to (1), and (4) to (2), system (b) is equivalent to system (a); hence, the one and only solution of system (a) is 3, 2.

Proof of (i). Let (1) and (2) be a system of equations in two unknowns,

$$\left.\begin{matrix} A = B, & (1) \\ C = D, & (2) \end{matrix}\right\}(a) \qquad \left.\begin{matrix} A' = B', & (3) \\ C' = D' & (4) \end{matrix}\right\}(b)$$

and let (3) be equivalent to (1), and (4) to (2); we are to prove that system (*b*) is equivalent to system (*a*).

Since (1) and (3) have the same solutions, and (2) and (4) also have the same solutions; it follows that any solution common to (1) and (2) will be common to (3) and (4) also; and conversely. Hence, systems (*a*) and (*b*) are equivalent.

202. The method of elimination by substitution depends upon the following principle of equivalence of systems:

(ii) **Substitution.** *If one equation of a system is solved for one of its unknowns, and the value thus obtained is substituted for this unknown in the other equation (or equations) of the system, the derived system will be equivalent to the given one.*

Ex. 1. Solve the system

$$\left.\begin{matrix} 2x = 10, & (1) \\ y = 12 - 13x. & (2) \end{matrix}\right\}(a)$$

From (1), $x = 5$.
Substituting this value of x in (2), we obtain
$y = 12 - 15 = -3$. } (*b*)

By (ii), system (*b*) is equivalent to system (*a*).
Hence, the one and only solution of system (*b*) or (*a*) is 5, -3.

Ex. 2. Solve the system

$$\left.\begin{matrix} 3x + 5y = 19, & (1) \\ 5x - 4y = 7. & (2) \end{matrix}\right\}(a)$$

From (1), $x = (19 - 5y)/3$. (3)
Substituting this value for x in (2), we have
$(5/3)(19 - 5y) - 4y = 7$. (4) } (*b*)

By (ii), system (b) is equivalent to system (a).

$$\left.\begin{array}{lr} \text{From (4),} \qquad y = 2. & (5) \\ \text{Substituting 2 for } y \text{ in (3), we obtain} & \\ \qquad x = (19 - 10)/3 = 3. & (6) \end{array}\right\} (c)$$

By (ii), system (c) is equivalent to system (b); hence, the one and only solution of system (c) or its equivalent system (a) is 3, 2.

Ex. 3. Solve the system

$$\left.\begin{array}{lr} 2x - 5y = 1, & (1) \\ 7x + 3y = 24. & (2) \end{array}\right\} (a)$$

$$\left.\begin{array}{lr} \text{From (1),} \qquad x = (5y + 1)/2. & (3) \\ \text{Substituting this value of } x \text{ in (2), we have} & \\ \qquad (7/2)(5y + 1) + 3y = 24. & (4) \end{array}\right\} (b)$$

By (ii), system (b) is equivalent to system (a).

$$\left.\begin{array}{lr} \text{From (4),} \qquad y = 1. & (5) \\ \text{Substituting this value of } y \text{ in (3), we obtain} & \\ \qquad x = (5 + 1)/2 = 3. & (6) \end{array}\right\} (c)$$

By (ii), system (c) is equivalent to (b), and therefore to (a). Hence, the one and only solution of system (a) is 3, 1.

The foregoing examples illustrate the following rule for eliminating by substitution.

From one of the equations find the value of the unknown to be eliminated, in terms of the others; then substitute this value for that unknown in the other equation or equations.

Proof of (ii). Let (1) and (2) be a system in two unknowns, and let (3) be the equation obtained by solving

$$\left.\begin{array}{lr} A = B, & (1) \\ C = D, & (2) \end{array}\right\} (a) \qquad \left.\begin{array}{lr} x = F, & (3) \\ C' = D', & (4) \end{array}\right\} (b)$$

(1) for x, and (4) the equation obtained by substituting for x in (2) its value F as given in (3); we are to prove that system (b) is equivalent to system (a).

Since $x = F$ is equivalent to equation (1), system (c) is by (i) equivalent to system (a),

$$\left.\begin{aligned} x &= F, &\quad (5)\\ C &= D. &\quad (6)\end{aligned}\right\} (c)$$

Any solution of system (c) renders $x \equiv F$ and $C \equiv D$; hence, any such solution must satisfy (6) after F has been substituted for x (vi, § 32); therefore, any solution of system (c) will be a solution of system (b).

Conversely, any solution of system (b) renders $x \equiv F$ and $C' \equiv D'$; hence, any such solution must satisfy (4) after x has been substituted for F (vi, § 32); hence, any solution of system (b) is a solution of (c).

Hence, system (b) is equivalent to system (c) or (a).

In like manner the theorem could be proved if the systems (a) and (b) contained three or more equations.

Exercise 79.

Solve each of the following systems of equations by the method of substitution:

1. $\left.\begin{aligned} 3x &= 27,\\ 2x + 3y &= 24.\end{aligned}\right\}$

2. $\left.\begin{aligned} 3x + 4y &= 58,\\ 2y &= 14.\end{aligned}\right\}$

3. $\left.\begin{aligned} 3x + 4y &= 10,\\ 4x + y &= 9.\end{aligned}\right\}$

4. $\left.\begin{aligned} x + 2y &= 13,\\ 3x + y &= 14.\end{aligned}\right\}$

5. $\left.\begin{aligned} 4x + 7y &= 29,\\ x + 3y &= 11.\end{aligned}\right\}$

6. $\left.\begin{aligned} 5x + 6y &= 17,\\ 6x + 5y &= 16.\end{aligned}\right\}$

7. $\left.\begin{aligned} 8x - y &= 34,\\ x + 8y &= 53.\end{aligned}\right\}$

8. $\left.\begin{aligned} 6y - 5x &= 18,\\ 12x - 9y &= 0.\end{aligned}\right\}$

9. $\left.\begin{aligned} 7x + 4y &= 1,\\ 9x + 4y &= 3.\end{aligned}\right\}$

10. $\left.\begin{aligned} x - 11y &= 1,\\ 111y - 9x &= 99.\end{aligned}\right\}$

11. $\left.\begin{aligned} 3x + 5y &= 19,\\ 5x - 4y &= 7.\end{aligned}\right\}$

12. $\left.\begin{aligned} 8x - 21y &= 5,\\ 6x + 14y &= -26.\end{aligned}\right\}$

13. $\left.\begin{array}{l} 3x - 11y = 0, \\ 19x - 19y = 8. \end{array}\right\}$

14. $\left.\begin{array}{l} 8x - 11y = 0, \\ 25x - 17y = 139. \end{array}\right\}$

15. $\left.\begin{array}{l} \dfrac{x}{3} - \dfrac{y}{6} = \dfrac{1}{2}, \\ \dfrac{x}{5} - \dfrac{3y}{10} = \dfrac{1}{2}. \end{array}\right\}$

16. $\left.\begin{array}{l} \frac{1}{3}x + 3y + 14 = 0, \\ \frac{1}{5}x + 5y + 4 = 0. \end{array}\right\}$

17. $\left.\begin{array}{l} \frac{1}{3}(7 + x) = \frac{1}{5}(9 + y), \\ \frac{1}{7}(11 + x + y) = \frac{1}{5}(9 + y). \end{array}\right\}$

18. $\left.\begin{array}{l} \frac{1}{2}(x + 1) = \frac{1}{3}(y + 2), \\ \frac{1}{4}(x + y) = \frac{1}{3}(y + 2). \end{array}\right\}$

19. $\left.\begin{array}{l} (x + 1)(y + 5) = (x + 5)(y + 1), \\ xy + x + y = (x + 2)(y + 2). \end{array}\right\}$

20. $\left.\begin{array}{l} xy - (x - 1)(y - 1) = 6(y - 1), \\ x - y = 1. \end{array}\right\}$

203. The following example illustrates a special form of the method of elimination by substitution, which is called **elimination by comparison.**

Ex. Solve the system

$$\left.\begin{array}{lr} 2x - 3y = 1, & (1) \\ 5x + 2y = 126. & (2) \end{array}\right\} (a)$$

Solve (1) for x, $\quad x = (3y + 1)/2. \quad (3)$

Solve (2) for x, $\quad x = (126 - 2y)/5. \quad (4)$

$$\text{(3), (4)} \; (b)$$

Substituting in (4) the value of x given in (3), or, what amounts to the same thing, putting these two values of x equal to each other, we obtain

$$\frac{3y + 1}{2} = \frac{126 - y}{5}, \text{ or } y = 13.$$

Substituting in (3), $\quad x = 40/2 = 20. \quad (c)$

By principles (i) and (ii), systems (a) and (c) are equivalent; or, in other words, no solution has been either lost or introduced in passing from system (a) to system (c); hence, the one and only solution of system (a) is 20, 13.

Exercise 80.

Solve each of the following systems by the method of comparison:

1. $\left.\begin{array}{l}\frac{2x}{3}+y=16,\\ x+\frac{y}{4}=14.\end{array}\right\}$

2. $\left.\begin{array}{l}x-y=5,\\ \frac{x}{4}-\frac{y}{5}=2.\end{array}\right\}$

3. $\left.\begin{array}{l}\frac{x}{9}+\frac{y}{7}=10,\\ \frac{1}{3}x+y=50.\end{array}\right\}$

4. $\left.\begin{array}{l}x=3y,\\ \frac{1}{3}x+y=34.\end{array}\right\}$

5. $\left.\begin{array}{l}\frac{5x}{6}-y=3,\\ x-\frac{5y}{6}=8.\end{array}\right\}$

6. $\left.\begin{array}{l}\frac{x}{5}+\frac{y}{2}=5,\\ x-y=4.\end{array}\right\}$

7. $\left.\begin{array}{l}\frac{2}{5}x-\frac{1}{12}y=3,\\ 4x-y=20.\end{array}\right\}$

8. $\left.\begin{array}{l}\frac{x}{3}+\frac{y}{4}=0,\\ 3x-7y=37.\end{array}\right\}$

9. $\left.\begin{array}{l}\frac{x+1}{10}=\frac{3y-5}{2},\\ \frac{x+1}{10}=\frac{x-y}{8}.\end{array}\right\}$

10. $\left.\begin{array}{l}\frac{x+3}{5}=\frac{8-y}{4},\\ \frac{3(x+y)}{8}=\frac{x+3}{5}.\end{array}\right\}$

204. The method of elimination by addition or subtraction depends upon the following principle:

(iii) **Addition.** *If an equation obtained by adding, or subtracting, the corresponding members of two or more equations of a system is put in the place of any one of these equations, the derived system will be equivalent to the given system.*

Ex. 1. Solve the system

$$\left.\begin{array}{lr}3x+7y=27, & (1)\\ 5x+2y=16. & (2)\end{array}\right\}(a)$$

To eliminate x, we obtain from (1) and (2) equivalent equations in which the coefficients of x are equal.

Multiply (1) by 5, $\quad 15x+35y=135, \quad (3)$

Multiply (2) by 3, $\quad 15x+6y=48. \quad (4)$

$$\left.\begin{array}{r}(3)\\(4)\end{array}\right\}(b)$$

$$\left.\begin{array}{lrr} \text{Subtract (4) from (3),} & 29y = 87, & (5) \\ \text{From (4),} & 15x + 6y = 48. & (6) \end{array}\right\} (c)$$

$$\left.\begin{array}{lr} \text{From (5),} & y = 3, \\ \text{Substitute in (6),} & x = 2. \end{array}\right\} (d)$$

Proof of equivalency. By (i), systems (a) and (b) are equivalent; by (iii), system (c) is equivalent to (b); and by (ii) and (i), system (d) is equivalent to (c).

Hence the one and only solution of (a) is 2, 3.

Ex. 2. Solve the system

$$\left.\begin{array}{rr} 7x + 2y = 47, & (1) \\ 5x - 4y = 1. & (2) \end{array}\right\} (a)$$

To eliminate y we obtain from (1) and (2) equivalent equations in which the coefficients of y are arithmetically equal.

$$\left.\begin{array}{lrr} \text{Multiply (1) by 2,} & 14x + 4y = 94. & (3) \\ \text{From (2),} & 5x - 4y = 1. & (4) \end{array}\right\} (b)$$

$$\left.\begin{array}{lrr} \text{Add (3) and (4),} & 19x = 95, \text{ or } x = 5. & (5) \\ \text{Substitute in (4),} & y = 6. & \end{array}\right\} (c)$$

Proof of equivalency. By (i), system (b) is equivalent to (a); by (iii), system (4) and (5) is equivalent to (b); and by (ii) and (i), system (c) is equivalent to system, (4) and (5).

Ex. 3. Solve the system

$$\left.\begin{array}{rr} (x-1)(y-2)-(x-2)(y-1) = -2, & (1) \\ (x+2)(y+2)-(x-2)(y-2) = 32. & (2) \end{array}\right\} (a)$$

We first reduce (1) and (2) to the form $ax + by = c$.

$$\left.\begin{array}{lrr} \text{From (1),} & x - y = 2, & (3) \\ \text{From (2),} & x + y = 8. & (4) \end{array}\right\} (b)$$

$$\begin{array}{lrr} \text{Add (3) and (4),} & 2x = 10, \text{ or } x = 5. & (5) \\ \text{Subtract (3) from (4),} & 2y = 6, \quad \text{or } y = 3. & (6) \end{array}$$

By (i), system (b) is equivalent to (a); by (i) and (iii), either (5) and (4) or (6) and (4) form a system equivalent to (b); hence the solution of (a) is given in (5) and (6).

Ex. 4. Solve the system

$$\left.\begin{aligned} 3x - \frac{y-5}{7} &= \frac{4x-3}{2}, &\quad (1)\\ \frac{3y+4}{5} - \frac{2x-5}{3} &= y. &\quad (2)\end{aligned}\right\}(a)$$

We first reduce (1) and (2) to the form $ax + by = c$.

$$\left.\begin{aligned} &\text{From (1),} &\quad 14x - 2y &= -31, &\quad (3)\\ &\text{From (2),} &\quad 10x + 6y &= 37. &\quad (4)\end{aligned}\right\}(b)$$

Multiply (3) by 3, $\quad 42x - 6y = -93. \quad (5)$

$$\left.\begin{aligned} &\text{Add (4) and (5),} &\quad 52x &= -56, \text{ or } x = -14/13.\\ &\text{Substitute in (3),} &\quad y &= 207/26.\end{aligned}\right\}(c)$$

Since (3) and (5) are equivalent and (3) is the simpler equation, to find y we substitute in (3) rather than in (5).

By (i), (ii), and (iii), no solution has been lost or introduced in passing from system (a) to system (c).

Proof of (iii). Let (1) and (2) be a system in two unknowns,

$$\left.\begin{aligned} A &= B, &\quad (1)\\ C &= D, &\quad (2)\end{aligned}\right\}(a) \qquad \left.\begin{aligned} A &= B, &\quad (3)\\ A + C &= B + D, &\quad (4)\end{aligned}\right\}(b)$$

and let (4) be obtained by adding the corresponding members of (1) and (2); we are to prove that system (b) is equivalent to system (a).

Any solution of (a) renders (1) and (2) identities. But, if (1) and (2) are identities; by § 32, (3) and (4) are identities; hence any solution of system (a) is a solution of system (b).

Conversely, any solution of (b) renders (3) and (4) identities. But, if (3) and (4) are identities; by § 32, (1) and (2) are identities; hence any solution of system (b) is a solution of (a).

Hence (a) and (b) are equivalent systems.

In like manner the theorem could be proved if the systems (a) and (b) contained three or more equations.

The foregoing examples illustrate the following rule for elimination by addition or subtraction:

Reduce the equations to the form $ax + by = c$.

Find the L. C. M. of the coefficients of the unknown to be eliminated. Multiply both members of each equation by the quotient of this L. C. M. divided by the coefficients of that unknown in the equation.

Add or subtract the corresponding members of the equations thus derived, according as the coefficients of the unknown to be eliminated are opposite or like in quality.

Exercise 81.

Solve each of the following systems by the method of addition or subtraction:

1. $\left.\begin{array}{l} 3x + 4y = 10, \\ 4x + y = 9. \end{array}\right\}$

2. $\left.\begin{array}{l} x + 2y = 13, \\ 3x + y = 14. \end{array}\right\}$

3. $\left.\begin{array}{l} 2x - y = 9, \\ 3x - 7y = 19. \end{array}\right\}$

4. $\left.\begin{array}{l} 4x + 7y = 29, \\ x + 3y = 11. \end{array}\right\}$

5. $\left.\begin{array}{l} 2x + y = 10, \\ 7x + 8y = 53. \end{array}\right\}$

6. $\left.\begin{array}{l} 5x + 6y = 17, \\ 6x + 5y = 16. \end{array}\right\}$

7. $\left.\begin{array}{l} 8x - y = 34, \\ x + 8y = 53. \end{array}\right\}$

8. $\left.\begin{array}{l} 15x + 7y = 29, \\ 9x + 15y = 39. \end{array}\right\}$

9. $\left.\begin{array}{l} 14x - 3y = 39, \\ 6x + 17y = 35. \end{array}\right\}$

10. $\left.\begin{array}{l} 28x - 23y = 33, \\ 63x - 25y = 101. \end{array}\right\}$

11. $\left.\begin{array}{l} 35x + 17y = 86, \\ 56x - 13y = 17. \end{array}\right\}$

12. $\left.\begin{array}{l} 5x - 7y = 0, \\ 7x + 5y = 74. \end{array}\right\}$

13. $\left.\begin{array}{l} 15x + 77y = 92, \\ 55x - 33y = 22. \end{array}\right\}$

14. $\left.\begin{array}{l} 5x = 7y - 21, \\ 21x - 9y = 75. \end{array}\right\}$

15. $\left.\begin{array}{l} 6y - 5x = 18, \\ 12x - 9y = 0. \end{array}\right\}$

16. $\left.\begin{array}{l} 21x - 50y = 60, \\ 28x - 27y = 199. \end{array}\right\}$

205. The following example illustrates a special form of the method of elimination by addition, which is called **elimination by undetermined multipliers.**

Ex. Solve the system

$$\left.\begin{aligned} 3x - 5y &= 2, \quad (1)\\ 5x - 2y &= 16. \quad (2)\end{aligned}\right\} (a)$$

Multiplying (1) by an arbitrary multiplier k, we obtain

$$3kx - 5ky = 2k. \quad (3)$$

Adding (3) and (2), we obtain

$$(3k + 5)x - (5k + 2)y = 2k + 16. \quad (4)$$

Putting the coefficient of x in (4) equal to 0, we obtain $k = -5/3$.

Substituting $-5/3$ for k in (4), we obtain $y = 2$. (5)

Putting the coefficient of y in (4) equal to 0, we obtain $k = -2/5$.

Substituting $-2/5$ for k in (4), we obtain $x = 4$. (6)

The two equations, $y = 2$ and $x = 4$, which result from the two different values of k in (4) form a system which by (i) and (iii) is equivalent to (a).

No one method of elimination is preferable for all cases. The learner should aim to select that method which is best suited to the system to be solved.

Exercise 82.

Solve each of the following systems by that method which is best suited to it:

1. $\left.\begin{aligned}57x + 25y &= 3772,\\ 25x + 57y &= 1148.\end{aligned}\right\}$

2. $\left.\begin{aligned}93x + 15y &= 123,\\ 15x + 93y &= 201.\end{aligned}\right\}$

3. $\left.\begin{aligned}15x + 19y &= 18,\\ 19x + 15y &= 50.\end{aligned}\right\}$

4. $\left.\begin{aligned}\frac{x}{2} + \frac{y}{3} &= 1,\\ \frac{x}{4} - \frac{2y}{3} &= 3.\end{aligned}\right\}$

5. $\left.\begin{aligned}\frac{x}{5} + 5y &= -4,\\ \frac{y}{5} + 5x &= 4.\end{aligned}\right\}$

6. $\left.\begin{aligned}\frac{x+2}{3}+4y=2,\\ \frac{y+11}{11}-\frac{x+1}{2}=1.\end{aligned}\right\}$

7. $\left.\begin{aligned}\frac{2x+3y}{5}+\frac{y+6}{7}=2,\\ \frac{2x-5y}{3}+\frac{x+7}{4}=1.\end{aligned}\right\}$

8. $\left.\begin{aligned}\frac{x-2}{3}-\frac{y+2}{4}=0,\\ \frac{2x-5}{5}-\frac{11-2y}{7}=0.\end{aligned}\right\}$

9. $\left.\begin{aligned}\frac{x-2}{3}-\frac{y+5}{2}=0,\\ \frac{2x-7}{3}-\frac{13-y}{16}=0.\end{aligned}\right\}$

10. $\left.\begin{aligned}\frac{x}{2}-\frac{y}{5}=4,\\ \frac{x}{7}+\frac{y}{15}=3.\end{aligned}\right\}$

11. $\left.\begin{aligned}2x+y=0,\\ \tfrac{1}{2}y-3x=8.\end{aligned}\right\}$

12. $\left.\begin{aligned}\tfrac{1}{7}x+\tfrac{1}{5}y=1\tfrac{3}{7},\\ x+\tfrac{1}{3}y=4\tfrac{2}{3}.\end{aligned}\right\}$

13. $\left.\begin{aligned}3x-7y=0,\\ \tfrac{2}{7}x+\tfrac{5}{3}y=7.\end{aligned}\right\}$

14. $\left.\begin{aligned}\tfrac{1}{5}x-\tfrac{1}{4}y=0,\\ 3x+\tfrac{1}{2}y=17.\end{aligned}\right\}$

15. $\left.\begin{aligned}ax+by=(a+b)^2,\\ ax-by=a^2-b^2.\end{aligned}\right\}$

16. $\left.\begin{aligned}ax+by=a^2+b^2,\\ bx+ay=2ab.\end{aligned}\right\}$

17. $\left.\begin{aligned}ax+by=a^2-b^2,\\ bx+ay=a^2-b^2.\end{aligned}\right\}$

18. $\left.\begin{aligned}x+y=a+b,\\ ax-by=b^2-a^2.\end{aligned}\right\}$

19. $\left.\begin{aligned}b^2x-a^2y=0,\\ bx+ay=a+b.\end{aligned}\right\}$

20. $\left.\begin{aligned}x-y=a-b,\\ ax-by=2a^2-2b^2.\end{aligned}\right\}$

21. $\left.\begin{aligned}ax-by=a^2+b^2,\\ x+y=2a.\end{aligned}\right\}$

22. $\left.\begin{aligned}bx-ay=b^3,\\ ax-by=a^3.\end{aligned}\right\}$

23. $\left.\begin{aligned}ax+by=1,\\ bx+ay=1.\end{aligned}\right\}$

24. $\left.\begin{aligned}(a+b)x-(a-b)y=3ab,\\ (a+b)y-(a-b)x=ab.\end{aligned}\right\}$

25. $\left.\begin{aligned}a^2x+b^2y=c^2,\\ a^3x+b^3y=c^3.\end{aligned}\right\}$

26. $\left.\begin{aligned}\frac{x}{a}+\frac{y}{b}=\frac{1}{ab},\\ \frac{x}{a'}-\frac{y}{b'}=\frac{1}{a'b'}.\end{aligned}\right\}$

27. $\left.\begin{aligned}\frac{3x}{a}+\frac{2y}{b}&=3,\\ \frac{9x}{a}-\frac{6y}{b}&=3.\end{aligned}\right\}$

28. $\left.\begin{aligned}qx-rb&=p(a-y),\\ \frac{qx}{a}+r&=p\left(1+\frac{y}{b}\right).\end{aligned}\right\}$

29. $\left.\begin{aligned}\frac{x}{m}+\frac{y}{m'}&=1,\\ \frac{x}{m'}-\frac{y}{m}&=1.\end{aligned}\right\}$

30. $\left.\begin{aligned}\frac{x}{a}+\frac{y}{b}&=1,\\ \frac{x}{3a}+\frac{y}{6b}&=\frac{2}{3}.\end{aligned}\right\}$

31. $\left.\begin{aligned}\frac{x}{a}+\frac{y}{b}&=2,\\ \frac{x}{a'}&=\frac{y}{b'}.\end{aligned}\right\}$

32. $\left.\begin{aligned}\frac{x}{a}-\frac{y}{b}&=1,\\ \frac{x}{b}+\frac{y}{a}&=\frac{a}{b}.\end{aligned}\right\}$

206. Two conditions are said to be *consistent* or *inconsistent* according as they *can* or *cannot* be satisfied at the same time.

Equations which express *consistent* conditions and therefore have one or more solutions in common are called **consistent equations.** Thus the equations in any of the above systems are consistent equations.

Equations which express *inconsistent* conditions and therefore have no solution in common are called **inconsistent equations.**

E.g., the equations $x + y = 4$, (1)

$$3x + 3y = 15, \tag{2}$$

express inconsistent conditions and have no solution in common. For if $x + y$ is 4, $3(x + y)$ is 12 and cannot therefore be 15.

207. Each of the foregoing systems of linear equations illustrates the following theorem:

If the two equations of a system in two unknowns are linear, independent, and consistent, the system has one, and only one, solution.

Proof. By the principles of equivalent equations and (i) in § 201, any system of two linear equations can be reduced to an *equivalent* system of the form

$$\left.\begin{aligned} ax + by &= c, &\quad (1)\\ a'x + b'y &= c'. &\quad (2)\end{aligned}\right\}(a)$$

Multiply (1) by b', $\qquad ab'x + bb'y = b'c. \qquad (3)$

Multiply (2) by b, $\qquad a'bx + bb'y = bc'. \qquad (4)$

Subtract (4) from (3), $\quad (ab' - a'b)x = b'c - bc'. \qquad (5)$ $\quad (b)$

From (1), $\qquad ax + by = c. \qquad (1)$

By (i) and (iii), system (b) is equivalent to system (a).

When $ab' - a'b = 0$ and $b'c - bc' = 0$, (5) is an identity, and system (b) or (a) has all the solutions of equation (1); hence equations (1) and (2) are *equivalent.*

When $ab' - a'b = 0$ and $b'c - bc' \neq 0$, no value of x will satisfy (5); hence system (b) or (a) has no solution, and equations (1) and (2) are *inconsistent.*

Hence (1) and (2) are not independent and consistent unless

$$ab' - a'b \neq 0.$$

When $ab' - a'b \neq 0$, x has one, and only one, value in (5), and this value of x will give one, and only one, value for y in (1); hence system (b) or (a) has one, and only one, solution.

When $ab' - a'b \neq 0$, from (5) we obtain

$$x = (b'c - bc') \,/\, (ab' - a'b). \qquad (6)$$

Similarly, $\qquad y = (ac' - a'c) \,/\, (ab' - a'b). \qquad (7)$

208. Systems of three linear equations.

Ex. 1. Solve

$$\left.\begin{aligned} 6x + 2y - 5z &= 13, &\quad (1)\\ 3x + 3y - 2z &= 13, &\quad (2)\\ 7x + 5y - 3z &= 26. &\quad (3)\end{aligned}\right\}(a)$$

To eliminate y, we can proceed as follows:

$$\left.\begin{array}{lrl} \text{Multiply (1) by 3,} & 18x + 6y - 15z = 39. & \\ \text{Multiply (2) by 2,} & 6x + 6y - 4z = 26. & \\ \text{Subtract,} & 12x - 11z = 13. & (4) \\ \text{Multiply (1) by 5,} & 30x + 10y - 25z = 65. & \\ \text{Multiply (3) by 2,} & 14x + 10y - 6z = 52. & \\ \text{Subtract,} & 16x - 19z = 13. & (5) \end{array}\right\} (b)$$

Solving system (b), *i.e.*, (4) and (5), we obtain

$$\left.\begin{array}{rr} z = 1, & (6) \\ x = 2. & (7) \end{array}\right\} (c)$$

From (6), (7), and (1), $\qquad y = 3. \qquad (8)$

The systems (b) and (c) are equivalent.

But (b) with (1) forms a system equivalent to (a); hence (c) with (1), or (c) with (8), forms a system equivalent to (a).

Hence the solution of system (a) is 2, 3, 1.

Ex. 2. Solve

$$\left.\begin{array}{rr} 3x + 2y + 4z = 19, & (1) \\ 2x + 5y + 3z = 21, & (2) \\ 3x - y + z = 4. & (3) \end{array}\right\} (a)$$

From (3), $\qquad y = 3x + z - 4. \qquad (4)$

Substituting in (1) and (2) the value of y in (4), we obtain

$$3x + 2(3x + z - 4) + 4z = 19,$$

and $\qquad 2x + 5(3x + z - 4) + 3z = 21;$

$$\left.\begin{array}{lr} \text{or,} & 9x + 6z = 27, \\ \text{and} & 17x + 8z = 41. \end{array}\right\} (b)$$

Solving system (b), we obtain

$$\left.\begin{array}{rr} z = 3, & (5) \\ x = 1. & (6) \end{array}\right\} (c)$$

From (4), (5), (6), $\qquad y = 2. \qquad (7)$

By (ii), system (b) with (4) forms a system equivalent to (a); hence (c) with (4), or (c) with (7), forms a system equivalent to (a). Hence the solution of (a) is 1, 2, 3.

The foregoing examples illustrate the following method of solving a system of three linear equations:

From any two of the three equations derive an equation, eliminating an unknown; next from the third equation and one of the other two derive a second equation eliminating the same unknown.

Solve for these two unknowns the two equations thus derived, and substitute the values of these two unknowns in the simplest equation which contains the third unknown.

209. From a *system of four linear equations* we can eliminate one of the four unknowns, and thus obtain a new system with three unknowns. Solving this new system, we can substitute the values thus obtained in the simplest equation which contains the fourth unknown.

Exercise 83.

Solve:

1. $\left.\begin{array}{l} x + 3y + 4z = 14, \\ x + 2y + z = 7, \\ 2x + y + 2z = 2. \end{array}\right\}$

2. $\left.\begin{array}{l} x + 2y + 2z = 11, \\ 2x + y + z = 7, \\ 3x + 4y + z = 14. \end{array}\right\}$

3. $\left.\begin{array}{l} 3x - 2y + z = 2, \\ 2x + 3y - z = 5, \\ x + y + z = 6. \end{array}\right\}$

4. $\left.\begin{array}{l} x + y + z = 1, \\ 2x + 3y + z = 4, \\ 4x + 9y + z = 16. \end{array}\right\}$

5. $\left.\begin{array}{l} 5x + 3y + 7z = 2, \\ 2x - 4y + 9z = 7, \\ 3x + 2y + 6z = 3. \end{array}\right\}$

6. $\left.\begin{array}{l} x + 2y - 3z = 6, \\ 2x + 4y - 7z = 9, \\ 3x - y - 5z = 8. \end{array}\right\}$

7. $\left.\begin{array}{l} x - 2y + 3z = 2, \\ 2x - 3y + z = 1, \\ 3x - y + 2z = 9. \end{array}\right\}$

8. $\left.\begin{array}{r} 3x + 2y - z = 20, \\ 2x + 3y + 6z = 70, \\ x - y + 6z = 41. \end{array}\right\}$

9. $\left.\begin{aligned} 2x+3y+4z&=20,\\ 3x+4y+5z&=26,\\ 3x+5y+6z&=31.\end{aligned}\right\}$

10. $\left.\begin{aligned} 3x-4y&=6z-16,\\ 4x-y&=z+5,\\ x&=3y+2(z-1).\end{aligned}\right\}$

11. $\left.\begin{aligned} ax+by&=1,\\ by+cz&=1,\\ cz+ax&=1.\end{aligned}\right\}$

12. $\left.\begin{aligned} cy+bz&=bc,\\ az+cx&=ca,\\ bx+ay&=ab.\end{aligned}\right\}$

13. $\left.\begin{aligned} x-\frac{y}{5}&=6,\\ y-\frac{z}{7}&=8,\\ z-\frac{x}{2}&=10.\end{aligned}\right\}$

14. $\left.\begin{aligned} \tfrac{1}{2}(x+z-5)&=y-z,\\ \tfrac{1}{2}(x+z-5)&=2x-11,\\ 2x-11&=9-(x+2z).\end{aligned}\right\}$

15. $\left.\begin{aligned} x+20&=\tfrac{3}{2}y+10,\\ x+20&=2z+5,\\ 2z+5&=110-(y+z).\end{aligned}\right\}$

16. $\left.\begin{aligned} ax+by&=1,\\ by+cz&=1,\\ cz+ax&=1.\end{aligned}\right\}$

17. $\left.\begin{aligned} cy+bz&=bc,\\ az+cx&=ca,\\ bx+cy&=ab.\end{aligned}\right\}$

18. $\left.\begin{aligned} x-ay+a^2z&=a^3,\\ x-by+b^2z&=b^3,\\ x-cy+c^2z&=c^3.\end{aligned}\right\}$

19. $\left.\begin{aligned} x+y+z-u&=11,\\ x+y-z+u&=17,\\ x-y+z+u&=9,\\ -x+y+z+u&=12.\end{aligned}\right\}$

20. $\left.\begin{aligned} x+y+z&=6,\\ x+y+u&=7,\\ x+z+u&=8,\\ y+z+u&=9.\end{aligned}\right\}$

SYSTEMS OF FRACTIONAL EQUATIONS.

210. In clearing of fractions the equations of a system, no solution will be lost, but new solutions may be introduced even when we clear of fractions in the simplest manner.

Ex. 1. Solve the system

$$\left.\begin{aligned} 4x-2y&=2, &(1)\\ \frac{5x+1}{3y-1}&=\frac{11}{8}. &(2)\end{aligned}\right\}\quad(a)$$

Clearing (2) of fractions and transposing, we have

$$40x-33y=-19. \qquad (3)$$

The solution of system, (1) and (3), is 2, 3.

In clearing (2) of fractions we multiplied by the unknown factor $3y-1$; hence any solution which was introduced will be a solution of the equation $3y-1=0$, or $3y+0x-1=0$.

Since 2, 3 is not a solution of this equation, it was not introduced in clearing (2) of fractions.

Hence 2, 3 is the one and only solution of system (a).

Ex. 2. Solve the system

$$\left.\begin{aligned} 5x-y&=2, &\quad(1)\\ \frac{1}{x-1}+\frac{1}{y-3}&=0. &\quad(2)\end{aligned}\right\}(a)$$

Clear (2) of fractions, $\quad x+y=4. \quad (3)$

The solution of system, (1) and (3), is 1, 3.

To clear (2) of fractions we multiplied by the unknown factor $(x-1)(y-3)$, and 1, 3 is a solution of the equation $(x-1)(y-3)=0$.

Hence the solution 1, 3 may have been introduced by clearing (2) of fractions.

By trial we find that 1, 3 is not a solution of (2); hence the solution 1, 3 was introduced, and system (a) has no solution; that is, its equations are inconsistent.

211. A system of fractional equations which are linear in the reciprocals of their unknowns is readily solved without clearing of fractions, by treating these reciprocals as the unknowns.

Ex. 1. Solve the system

$$\left.\begin{aligned} a/x+c/y&=m, &\quad(1)\\ b/x+d/y&=n. &\quad(2)\end{aligned}\right\}(a)$$

Multiply (1) by b, $\quad ab(1/x)+cb(1/y)=bm. \quad (3)$

Multiply (2) by a, $\quad ab(1/x)+ad(1/y)=an. \quad (4)$

Subtract (4) from (3), $\quad (bc-ad)(1/y)=bm-an.$

$$\therefore\ y=\frac{bc-ad}{bm-an}. \quad (5)$$

Multiply (1) by d, $\quad ad(1/x)+cd(1/y)=dm. \quad (6)$

Multiply (2) by c, $\quad bc(1/x)+cd(1/y)=cn. \quad (7)$

Subtract (7) from (6), $\quad (ad-bc)(1/x)=dm-cn.$

$$\therefore\ x=\frac{ad-bc}{dm-cn}. \quad (8)$$

Multiplying (1) and (2) by xy to clear them of fractions would give us a system of quadratic equations and introduce the new solution 0, 0.

Ex. 2. Solve the system

$$\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}+\frac{1}{z}&=36, & (1)\\ \frac{1}{x}+\frac{3}{y}-\frac{1}{z}&=28, & (2)\\ \frac{1}{x}+\frac{1}{3y}+\frac{1}{2z}&=20. & (3) \end{aligned}\right\} (a)$$

Subtract (1) from (2), $2(1/y)-2(1/z)=-8,$ (4)

Subtract (3) from (1), $\frac{2}{3}(1/y)+\frac{1}{2}(1/z)=16.$ (5)

(4), (5): (b)

Solving system (b), $y=1/12,$ (6)

$z=1/16.$ (7)

From (1), (6), and (7), $x=1/8.$ (8)

Exercise 84.

Solve the system

1. $\left.\begin{aligned}\frac{8}{x}-\frac{9}{y}&=1,\\ \frac{10}{x}+\frac{6}{y}&=7.\end{aligned}\right\}$

2. $\left.\begin{aligned}\frac{9}{x}-\frac{4}{y}&=2,\\ \frac{18}{x}+\frac{18}{y}&=10.\end{aligned}\right\}$

3. $\left.\begin{aligned}\frac{6}{x}-\frac{5}{y}&=9,\\ \frac{7}{x}-\frac{2}{y}&=5.\end{aligned}\right\}$

4. $\left.\begin{aligned}\frac{3}{x}+\frac{4}{y}&=3,\\ \frac{6}{x}-\frac{2}{y}&=1.\end{aligned}\right\}$

5. $\left.\begin{aligned}\frac{5}{x}+\frac{16}{y}&=79,\\ \frac{16}{x}-\frac{1}{y}&=44.\end{aligned}\right\}$

6. $\left.\begin{aligned}\frac{6}{x}-\frac{7}{y}&=2,\\ \frac{2}{x}+\frac{14}{y}&=3.\end{aligned}\right\}$

7. $\left.\begin{aligned}\frac{5}{3x}+\frac{2}{5y}&=7,\\ \frac{7}{6x}-\frac{1}{10y}&=3.\end{aligned}\right\}$

8. $\left.\begin{aligned}\frac{1}{2x}+\frac{2}{3y}&=3,\\ \frac{3}{4x}+\frac{4}{5y}&=3.9.\end{aligned}\right\}$

9. $\left.\begin{aligned} \frac{m}{x}+\frac{n}{y}&=a, \\ \frac{b}{x}+\frac{c}{y}&=d. \end{aligned}\right\}$

10. $\left.\begin{aligned} \frac{m}{x}-\frac{n}{y}&=\frac{mr}{n}, \\ \frac{n}{x}-\frac{m}{y}&=\frac{nr}{m}. \end{aligned}\right\}$

11. $\left.\begin{aligned} \frac{2}{mx}-\frac{1}{ny}&=3, \\ \frac{3}{mx}+\frac{4}{ny}&=7. \end{aligned}\right\}$

12. $\left.\begin{aligned} \frac{7}{ax}+\frac{3}{by}&=2, \\ \frac{5}{ax}-\frac{2}{by}&=7. \end{aligned}\right\}$

13. $\left.\begin{aligned} \frac{3}{ax}-\frac{2}{by}&=4, \\ \frac{b}{x}+\frac{a}{y}&=7. \end{aligned}\right\}$

14. $\left.\begin{aligned} \frac{m}{nx}+\frac{n}{my}&=m+n, \\ \frac{n}{x}+\frac{m}{y}&=m^2+n^2. \end{aligned}\right\}$

15. $\left.\begin{aligned} \frac{1}{x}-\frac{2}{y}+4&=0, \\ \frac{1}{y}-\frac{1}{z}+1&=0, \\ \frac{2}{z}+\frac{3}{x}&=14. \end{aligned}\right\}$

16. $\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}+\frac{1}{z}&=36, \\ \frac{1}{x}+\frac{3}{y}-\frac{1}{z}&=28, \\ \frac{1}{x}+\frac{1}{3y}+\frac{1}{2z}&=20. \end{aligned}\right\}$

17. $\left.\begin{aligned} \frac{1}{2x}+\frac{1}{4y}-\frac{1}{3z}&=\frac{1}{4}, \\ \frac{1}{x}&=\frac{1}{3y}, \\ \frac{1}{x}-\frac{1}{5y}+\frac{4}{z}&=2\tfrac{2}{15}. \end{aligned}\right\}$

18. $\left.\begin{aligned} 3x+4y&=11, \\ \frac{7x+4}{3y+1}&=\frac{11}{7}. \end{aligned}\right\}$

19. $\left.\begin{aligned} \frac{2x+4}{3y-1}&=1, \\ \frac{7x-2}{y+3}&=2. \end{aligned}\right\}$

20. $\left.\begin{aligned} \frac{x+2y+2}{3x+y-1}&=1, \\ \frac{3x+y+1}{4x-y-2}&=1. \end{aligned}\right\}$

21. $\left.\begin{aligned} \frac{1}{5x+2y}&=\frac{1}{9x+4y}, \\ \frac{2x}{7x+y}&=\frac{2}{5}. \end{aligned}\right\}$

22. $\left.\begin{aligned} 5x-3y&=3, \\ \frac{1}{x-3}+\frac{1}{y-4}&=0. \end{aligned}\right\}$

CHAPTER XV

PROBLEMS SOLVED BY SYSTEMS

212. A *determinate problem* is one which has a finite number of solutions. Every determinate problem must contain as many independent consistent conditions, expressed or implied, as unknown numbers. If in any such problem we denote each unknown by a letter, and express each condition by an equation, we shall obtain as many independent consistent equations as there are unknowns.

The solutions of the system of equations thus obtained will give the solutions of the problem.

Prob. 1. Find two numbers such that twice the greater exceeds three times the less by 6, and that twice the less exceeds the greater by 2.

Let $x =$ the greater number, and $y =$ the less.

Then, by the *first* condition, we have

$$2x - 3y = 6, \qquad (1)$$

and by the *second* condition we have

$$2y - x = 2. \qquad (2)$$

(a) [system of (1) and (2)]

From system (a), $x = 18$, the greater number;

and $y = 10$, the less number.

Prob. 2. A number expressed by two digits is equal to six times the sum of its digits, and the digit in the tens' place is greater by one than the digit in the units' place. Find the number.

Let $x =$ the digit in tens' place,

and $y =$ the digit in units' place.

Then, from the first condition, we have

$$10x + y = 6(x + y), \tag{1}$$

and from the second condition we have

$$x - y = 1. \tag{2}$$

From system (a), $x = 5$, the digit in tens' place;

and $y = 4$, the digit in units' place.

That is, the required number is 54.

Prob. 3. If the numerator of a fraction is increased by 2 and the denominator by 1, it becomes equal to 5/8, and if the numerator and denominator are each diminished by 1, it becomes equal to 1/2. Find the fraction.

Let $x =$ the numerator, and $y =$ the denominator; then,

from the first condition, $$\frac{x+2}{y+1} = \frac{5}{8}, \tag{1}$$

and from the second, $$\frac{x-1}{y-1} = \frac{1}{2}. \tag{2}$$

The solution of system (a) is 8, 15; hence the fraction is 8/15.

Prob. 4. A man and a boy can do in 15 days a piece of work which would be done in 2 days by 7 men and 9 boys. How long would it take one boy or one man to do it.

Let $x =$ the number of days it would take one man to do the whole work, and $y =$ the number of days it would take one boy.

Let the whole work be represented by 1.

Then in one day a man would do $1/x$ of the work, and a boy $1/y$ of it.

Hence, by the first condition, we have

$$15/x + 15/y = 1, \tag{1}$$

and by the second condition we have

$$14/x + 18/y = 1. \tag{2}$$

The solution of system (a) is 20, 60.

Hence one man would do the work in 20 days, and one boy in 60 days.

Exercise 85.

1. Six horses and 7 cows can be bought for $1250, and 13 cows and 11 horses can be bought for $2305. Find the value of each animal.

2. Four times B's age exceeds A's age by 20 years, and $\frac{1}{3}$ of A's age is less than B's age by 2 years. Find their ages.

3. Find a fraction such that if 1 be added to its denominator it reduces to $\frac{1}{2}$, and if 2 be added to its numerator it reduces to $\frac{3}{5}$.

4. A man being asked his age, replied: "If you take 2 years from my present age the result will be double my wife's age, and 3 years ago her age was $\frac{1}{3}$ of what mine will be in 12 years." Find their ages.

5. One-eleventh of A's age is greater by 2 years than $\frac{1}{7}$ of B's, and twice B's age is equal to what A's age was 13 years ago. Find their ages.

6. In 8 hours A walks 12 miles more than B does in 7 hours; and in 13 hours B walks 7 miles more than A does in 9 hours. How many miles does each walk per hour?

7. At an election the majority was 162, which was $\frac{3}{11}$ of the whole number of voters. What was the number of the votes on each side?

8. A and B have $250 between them; but if A were to lose half his money, and B $\frac{2}{3}$ of his, they would then have only $100. How much has each?

9. A man bought 8 cows and 50 sheep for $1125. He sold the cows at a profit of 20%, and the sheep at a profit of 10%, and received in all $1287.50. What was the cost of each cow and of each sheep?

10. Twenty-eight tons of goods are to be carried in carts and wagons, and it is found that this will require 15 carts and 12 wagons, or else 24 carts and 8 wagons. How much can each cart and each wagon carry?

11. A and B can perform a certain task in 30 days, working together. After 12 days, however, B was called off, and A finished it by himself 24 days after. How long would each take to do the work alone?

12. Find the fraction such that if you quadruple the numerator and add 3 to the denominator the fraction will be doubled, but if you add 2 to the numerator and quadruple the denominator, the fraction will be halved.

13. The first edition of a book had 600 pages, and was divided into two parts. In the second edition $\frac{1}{4}$ of the second part was omitted and 30 pages were added to the first part. The change made the two parts of the same length. How many pages were in each part in the first edition?

14. A marketman bought eggs, some at 3 for 5 cents, and some at 4 for 5 cents, and paid for all $5.60; he afterwards sold them at 24 cents a dozen, clearing $1.80. How many eggs did he buy at each price?

15. In a bag containing black and white balls, half the number of white is equal to a third of the number of black; and twice the whole number of balls exceeds 3 times the number of black balls by 4. How many balls does the bag contain?

16. A crew that can row 10 miles an hour down a river, finds that it takes twice as long to row up the river as to row down. Find the rate of the current.

17. A certain number between 10 and 100 is 8 times the sum of its digits, and if 45 be subtracted from it the digits will be reversed. Find the number.

18. If A were to receive $50 from B, he would then have twice as much as B would have left; but if B were to receive $50 from A, B would have 3 times as much as A would have left. How much has each?

19. A farmer sold 30 bushels of wheat and 50 bushels of barley for $93.75. He also sold at the same prices 50 bushels of wheat and 30 bushels of barley for $96.25. What was the price of the wheat per bushel?

20. One rectangle is of the same area as another which is 6 yards longer and 4 yards narrower; it is also of the same area as a third, which is 8 yards longer and 5 yards narrower. What is the area of each?

21. A boy rows 8 miles with the current in 1 hour 4 minutes, and returns against the current in $2\frac{2}{7}$ hours. At what rate would he row in still water? What is the rate of the current?

22. A, B, C, D have $1450 among them; A has twice as much as C, and B has 3 times as much as D; also C and D together have $250 less than A. Find how much each has.

23. A, B, C, D have $1350 among them; A has 3 times as much as C, and B 5 times as much as D; also A and B together have $250 less than 8 times what C has. Find how much each has.

24. A number consists of 2 digits followed by zero. If the digits be interchanged, the number will be diminished by 180; if the left-hand digit be halved, and the other digit be interchanged with zero, the number will be diminished by 454. Find the number.

25. A train travelled a certain distance at a uniform rate; had the speed been 6 miles an hour more, the journey would have occupied 4 hours less; and had the speed been 6 miles an hour less, the journey would have occupied 6 hours more. Find the distance.

Let $x =$ the number of miles the train runs per hour,

and $y =$ the number of hours the journey takes.

$$\left.\begin{aligned} \text{Then}\quad xy &= (x+6)(y-4), \\ \text{and}\quad xy &= (x-6)(y+6). \end{aligned}\right\}$$

26. A traveller walks a certain distance; had he gone $\frac{1}{2}$ mile an hour faster, he would have walked it in $\frac{4}{5}$ of the time; had he gone $\frac{1}{2}$ mile an hour slower, he would have been $2\frac{1}{2}$ hours longer on the road. Find the distance.

27. A man walks 35 miles, partly at the rate of 4 miles an hour, and partly at 5; if he had walked at 5 miles an hour when he walked at 4, and *vice versa*, he would have covered 2 miles more in the same time. Find the time he was walking.

28. A fishing-rod consists of two parts; the length of the upper part is $\frac{5}{7}$ that of the lower part; and 9 times the upper part together with 13 times the lower part exceeds 11 times the whole rod by 36 inches. Find the lengths of the two parts.

29. A man put \$12,000 at interest in three sums, the first at 5 per cent, the second at 4 per cent, and the third at 3 per cent, receiving for the whole \$490 a year. The sum at 5 per cent is half as much as the other two sums. Find each of the three sums.

30. A, B, and C can together do a piece of work in 30 days; A and B can together do it in 32 days; B and C can together do it in 120 days. Find the time in which each alone could do the work.

31. A certain company in a hotel found, when they came to pay their bills, that if there had been 3 more persons to pay the same bill, they would have paid \$1 each less than they did; and if there had been 2 fewer persons, they would have paid \$1 each more than they did. Find the number of persons, and the number of dollars each paid.

32. A railway train, after travelling 1 hour, is detained 30 minutes, after which it proceeds at $\frac{3}{4}$ of its former rate, and arrives 20 minutes late. If the detention had occurred 10 miles farther on, the train would have arrived 5 minutes later than it did. Find the first rate of the train, and the distance travelled.

Let $x =$ the number of miles the train at first ran per hour;

and $y =$ the number of miles in the whole distance travelled.

Then $y - x =$ the number of miles to be travelled after the detention,

$\frac{y-x}{x} =$ the number of hours required to travel $y - x$ miles at the rate before the detention,

and $\frac{4(y-x)}{5x} =$ the number of hours required to travel $y - x$ miles at the rate after the detention.

Hence $$\frac{y-x}{x} - \frac{4(y-x)}{5x} = \frac{10}{60}. \qquad (1)$$

Similarly, $$\frac{y-x-10}{x} - \frac{4(y-x-10)}{5x} = \frac{5}{60}. \qquad (2)$$

Subtract (2) from (1), $$\frac{10}{x} - \frac{40}{5x} = \frac{5}{60}.$$

$$\therefore\ x = 24,\ y = 44.$$

Hence the first rate was 24 miles an hour, and the distance travelled was 44 miles.

33. A railway train, after travelling 1 hour, meets with an accident which delays it 1 hour, after which it proceeds at $\frac{3}{5}$ of its former rate, and arrives at the terminus 3 hours behind time; had the accident occurred 50 miles farther on, the train would have arrived 1 hour 20 minutes sooner. Find the length of the line, and the original rate of the train. *Ans.* 100 miles, 25 miles per hour.

34. A jockey has 2 horses and 2 saddles. The saddles are worth \$15 and \$10 respectively. The value of the better horse and better saddle is $\frac{4}{3}$ that of the other horse and saddle; and the value of the better saddle and poorer horse is $\frac{13}{15}$ that of the other horse and saddle. Find the worth of each horse.

35. Five thousand dollars is divided among A, B, C, and D. B gets half as much as A; the excess of C's share over D's share is equal to $\frac{1}{3}$ of A's share, and if B's share

were increased by \$ 500 he would have as much as C and D have between them. Find how much each gets.

36. A party was composed of a certain number of men and women, and, when 4 of the women were gone, it was observed that there were left just half as many men again as women; they came back, however, with their husbands, and now there were only a third as many men again as women. What was the original number of each?

37. Two vessels contain mixtures of wine and water; in one there is 3 times as much wine as water, in the other 5 times as much water as wine. Find how much must be drawn off from each to fill a third vessel which holds 7 gallons, in order that its contents may be half wine and half water.

38. There is a number of 3 digits, the last of which is double the first; when the number is divided by the sum of the digits, the quotient is 22; and when by the product of the last two, 11. Find the number.

39. Some smugglers found a cave which would exactly hold the cargo of their boat; viz. 13 bales of silk and 33 casks of rum. While unloading, a revenue cutter came in sight, and they were obliged to sail away, having landed only 9 casks and 5 bales, and filled $\frac{1}{3}$ of the cave. How many bales separately, or how many casks, would it contain?

40. There are 2 alloys of silver and copper, of which one contains twice as much copper as silver, and the other 3 times as much silver as copper. How much must be taken from each to weigh a kilogram, of which the silver and the copper shall be equal in weight?

41. A person rows a distance of 20 miles, and back again, in 10 hours, the stream flowing uniformly in the same direction all the time; and he finds that he can row 2 miles against the stream in the same time that he rows 3 miles with it. Find the time of his going and returning.

42. A and B can do a piece of work in m days, A and C can do the same piece in n days, and B and C can do it in p days. Find in how many days each can do the work.

43. For $26.25 we can buy either 32 pounds of tea and 15 pounds of coffee, or 36 pounds of tea and 9 pounds of coffee. Find the price of a pound of each.

44. A pound of tea and 3 pounds of sugar cost $1.50; but if sugar were to rise 50 per cent, and tea 10 per cent, they would cost $1.75. Find the price of tea and sugar.

45. A person possesses a certain capital which is invested at a certain rate per cent. A second person has $5000 more capital than the first person, and invests it at 1 per cent more; thus his income exceeds that of the first person by $400. A third person has $7500 more capital than the first, and invests it at 2 per cent more; thus his income exceeds that of the first person by $750. Find the capital of each person and the rate at which it is invested.

46. Two plugs are opened in the bottom of a cistern containing 192 gallons of water; after 3 hours one of the plugs becomes stopped, and the cistern is emptied by the other in 11 more hours; had 6 hours occurred before the stoppage, it would have required only 6 hours more to empty the cistern. How many gallons will each plug-hole discharge in an hour, supposing the discharge uniform?

47. A certain number of persons were divided into 3 classes, such that the majority of the first and second classes together over the third was 10 less than 4 times the majority of the second and third together over the first; but if the first class had 30 more, and the second and third together 29 less, the first would have outnumbered the last 2 classes by 1. Find the number in each class when the whole number was 34 more than 8 times the majority of the third class over the second.

48. Two persons, A and B, could finish a work in m days; they worked together n days, when A was called off, and B finished it in p days. In what time could each do it?

49. The fore-wheel of a carriage makes 6 revolutions more than the hind-wheel in going 120 yards; if the circumference of the fore-wheel be increased by $\frac{1}{4}$ of its present size, and the circumference of the hind-wheel by $\frac{1}{5}$ of its present size, the 6 will be changed to 4. Required the circumference of each wheel.

CHAPTER XVI

EVOLUTION. IRRATIONAL NUMBERS

213. An ***n*th root** of a given number is a number whose nth power is equal to the given number.

E.g., one *second* root of 4 is 2, since $2^2 = 4$.
Another *second* root of 4 is -2, since $(-2)^2 = 4$.
A *third* root of -8 is -2, since $(-2)^3 = -8$.

A *second* root of a number is usually called a *square* root; and a *third* root a *cube* root.

214. The **radical sign**, $\sqrt{\ }$, written before a number, denotes a root of that number.

The **radicand** is the number whose root is required.

The **index** is the number which, written before and a little above the radical sign, indicates *what* root is required. When no index is written, 2 is understood.

E.g., $\sqrt[2]{16}$ or $\sqrt{16}$ denotes a *second*, or *square*, root of 16; 16 is the *radicand*, and 2 is the *index*.

The expression $\sqrt[n]{u}$ denotes *an nth root of* u; u is the *radicand*, and n the *index*.

215. Since by definition $(\sqrt[n]{u})^n \equiv u$, it follows that $\sqrt[n]{u}$ is one of the n equal factors of u.

216. A **rational**, or **commensurable**, **number** is any whole or fractional number.

A **rational expression** is one which *can* be written without using an indicated root. All the expressions in the previous chapters are rational expressions.

217. A **perfect *n*th power** is a number or expression whose nth root is a rational number or expression.

E.g., since $\sqrt{25} = 5$, 25 is a *perfect square.*
Since $\sqrt[3]{-8x^3y^6} \equiv -2xy^2$, $-8x^3y^2$ is a *perfect cube.*

Prior to § 238 each radicand will be a perfect power of a degree equal to the index of the root.

218. Two roots are said to be **like** or **unlike** according as their indices are equal or unequal.

An **even root** is one whose *index* is *even;* as, $\sqrt[4]{x^8}$.

An **odd root** is one whose *index* is *odd;* as, $\sqrt[3]{27}$.

219. Number of roots.

(i) *An arithmetic number has one, and only one,* ***n*th** *root.*

Any *odd* power of a positive or negative base has the same quality as the base itself; hence,

(ii) *A positive or a negative number has one odd root of the same quality as the number itself.*

E.g., one value of $\sqrt[3]{+27}$ is $+3$, since $(+3)^3 = +27$.
Again, one value of $\sqrt[5]{-32}$ is -2, since $(-2)^5 = -32$.

If two numbers, opposite in quality, are arithmetically equal, their like *even* powers are the same *positive* number; hence,

(iii) *A positive number has two even roots, which are arithmetically equal, and opposite in quality.*

E.g., two values of $\sqrt{+81}$ are $+9$ and -9, since $(+9)^2$ or $(-9)^2$ is $+81$.

Again, two values of $\sqrt[4]{+81}$ are $+3$ and -3, since $(+3)^4$ or $(-3)^4$ is $+81$.

Any *even* power of a positive or a negative number is *positive;* hence an *even* root of a *negative* number cannot be a positive or a negative number.

Even roots of *negative* numbers give rise to *new* quality-numbers, which will be considered in Chapter XVIII.

220. The **principal root** of a *positive* number is its *positive* root.

The **principal odd root** of a *negative* number is its *negative* root. *E.g.*, $+4$ is the principal square root of 16, and -3 is the principal cube root of -27.

Unless the contrary is stated, the radical sign will hereafter be understood as denoting only the *principal* root.

221. The like principal roots of equal numbers are equal; hence,

The like principal roots of identical expressions are identical expressions.

222. Evolution is the operation of finding any required root of a number or expression.

In the statement of the following principles of roots, by "the root" is meant "the principal root."

223. *The exponent of any base in the root is equal to the exponent of that base in the radicand divided by the index of the root; and conversely.*

That is, $$\sqrt[n]{a^{mn}} \equiv a^m. \quad (1)$$

Proof. By § 118, $(a^m)^n \equiv a^{mn}$.

Hence, by § 221, $a^m \equiv \sqrt[n]{a^{mn}}$, and conversely (1).

E.g., $\sqrt[3]{a^6} \equiv a^{6 \div 3} \equiv a^2$; $\sqrt[5]{x^{15}} \equiv x^3$.

224. *The **n**th root of a product is equal to the product of the **n**th roots of its factors; and conversely.*

That is, $$\sqrt[n]{ab} \equiv \sqrt[n]{a} \cdot \sqrt[n]{b}. \quad (1)$$

Proof. By § 119, $(\sqrt[n]{a} \cdot \sqrt[n]{b})^n \equiv (\sqrt[n]{a})^n (\sqrt[n]{b})^n \equiv ab$.

Hence, by § 221, $\sqrt[n]{a} \cdot \sqrt[n]{b} \equiv \sqrt[n]{ab}$, and conversely (1).

Ex. 1. $\sqrt[5]{-32\,a^{10}} \equiv \sqrt[5]{-32} \cdot \sqrt[5]{a^{10}} \equiv -2\,a^2$.

Ex. 2. $\sqrt[3]{-a^9b^6} \equiv \sqrt[3]{-1} \cdot \sqrt[3]{a^9} \cdot \sqrt[3]{b^6} \equiv -a^3b^2$.

Observe that, from this principle, it follows that the nth root of any real number is the nth root of its quality-unit into the nth root of its arithmetic value. Thus,

$$\sqrt[3]{+27} = \sqrt[3]{+1}\cdot\sqrt[3]{27} = +3;\quad \sqrt[5]{-32} = \sqrt[5]{-1}\cdot\sqrt[5]{32} = -2.$$

225. *The nth root of the quotient of two numbers is equal to the quotient of their nth roots; and conversely.*

That is, $$\sqrt[n]{a/b} \equiv \sqrt[n]{a}/\sqrt[n]{b}. \quad (1)$$

Proof. By § 186, $$\left(\frac{\sqrt[n]{a}}{\sqrt[n]{b}}\right)^n \equiv \frac{(\sqrt[n]{a})}{(\sqrt[n]{b})^n} \equiv \frac{a}{b}$$

Hence, by § 221, $\sqrt[n]{a}/\sqrt[n]{b} \equiv \sqrt[n]{a/b}$, and conversely (1).

Ex. 1. $$\sqrt[3]{\frac{-125\,x^6}{216\,a^{12}}} \equiv \frac{\sqrt[3]{-125\,x^6}}{\sqrt[3]{216\,a^{12}}} \qquad \text{§ 225}$$

$$\equiv (-5\,x^2)/(6\,a^4). \qquad \text{§ 224}$$

Ex. 2. $$\sqrt[5]{-\frac{32\,x^5y^{10}}{a^5b^{10}z^{15}}} \equiv \frac{\sqrt[5]{-32\,x^5y^{10}}}{\sqrt[5]{a^5b^{10}z^{15}}} \qquad \text{§§ 167, 225}$$

$$\equiv -2\,xy^2/(ab^2z^3). \qquad \text{§ 224}$$

Exercise 86.

Reduce to a rational form the following expressions:

1. $\sqrt{4\,a^2b^2}$.
2. $\sqrt{9\,x^6y^2}$.
3. $\sqrt{25\,a^4b^6}$.
4. $\sqrt{100\,a^8}$.
5. $\sqrt[3]{27\,a^6b^3}$.
6. $\sqrt[3]{-8\,x^9y^6}$.
7. $\sqrt[3]{-a^9x^6y^{12}}$.
8. $\sqrt[3]{-64\,x^3y^6}$.
9. $\sqrt[3]{343\,a^{12}b^6}$.
10. $\sqrt[4]{81\,a^8b^4}$.
11. $\sqrt[4]{16\,x^{12}y^8}$.
12. $\sqrt[5]{32\,x^5y^{10}}$.
13. $\sqrt{\dfrac{a^6b^4}{16}}$.
14. $\sqrt{\dfrac{400\,a^{10}b^6}{81\,x^8y^{12}}}$.
15. $\sqrt[3]{\dfrac{125\,a^3b^6}{216\,x^9y^6}}$.
16. $\sqrt[3]{-\dfrac{27\,x^6y^3}{64\,a^9b^{12}}}$.
17. $\sqrt[5]{\dfrac{32\,x^5y^{10}}{-a^{10}b^{15}}}$.
18. $\sqrt[7]{-\dfrac{128}{a^{42}b^{63}}}$.

19. $\sqrt{\frac{2}{3}-\frac{5}{9}}$. 21. $\sqrt{\frac{4}{5}+\frac{16}{25}}$. 23. $\sqrt{\frac{5}{6}-\frac{7}{18}}$.

20. $\sqrt{\frac{3}{4}-\frac{11}{16}}$. 22. $\sqrt{\frac{7}{8}-\frac{7}{64}}$. 24. $\sqrt{\frac{6}{7}-\frac{6}{49}}$.

226. *The* **s***th root of the* **r***th power of a number is equal to the* **r***th power of its* **s***th root; and conversely.*

That is, $\sqrt[s]{a^r} \equiv (\sqrt[s]{a})^r.$ (1)

Proof. Let $\sqrt[s]{a} \equiv B;$ (2)

then, $a \equiv B^s.$ § 128

$\therefore\ a^r \equiv (B^s)^r \equiv B^{rs},$ §§ 128, 118

Hence, by 221, $\sqrt[s]{a^r} \equiv B^r.$ (3)

From (2), $(\sqrt[s]{a})^r \equiv B^r.$ (4)

From (3) and (4), by § 32, we obtain (1).

Ex. 1. $\sqrt[3]{(64/125)^2} = (\sqrt[3]{64/125})^2$
$= (4/5)^2 = 16/25.$

Ex. 2. $\sqrt{(81\,x^{2n}c^4)^3} \equiv (\sqrt{81\,x^{2n}c^4})^3$
$\equiv (9\,x^nc^2)^3 \equiv 729\,x^{3n}c^6.$

227. *The* **s***th root of the* **q***th root of a number is equal to the* **qs***th root of the number; that is,*

$$\sqrt[s]{\sqrt[q]{a}} \equiv \sqrt[qs]{a}.$$

Proof. If a number is resolved into q equal factors, and then each one of these q equal factors is resolved into s equal factors, the number will be resolved into qs equal factors; that is, $\sqrt[s]{\sqrt[q]{a}} \equiv \sqrt[qs]{a}.$

Ex. 1. $\sqrt[3]{\sqrt{(2^6\,x^6y^{12})}} \equiv \sqrt[6]{(2^6\,x^6y^{12})} \equiv 2\,xy^2.$ § 227

Ex. 2. $\sqrt[3]{\sqrt{(25^3 \times 9^3)}} = \sqrt[6]{(5^6 \times 3^6)} = 5 \times 3 = 15.$

Exercise 87.

Reduce each of the following expressions to a rational form:

1. $\sqrt[3]{}\sqrt{}(64\,a^{12}b^6)$.
2. $\sqrt{}\sqrt[3]{}(27^2 \times 64^2)$.
3. $\sqrt{}\sqrt[3]{}(a^{12}b^{18}c^6x^{12})$.
4. $\sqrt{(16/49)^3}$.
5. $\sqrt[3]{(216\,a^{3n}/27)^2}$.
6. $\sqrt[3]{(27\,x^3y^6z^{12})^5}$.
7. $\sqrt[3]{\dfrac{.027\,x^9}{.064\,a^6}}$.
8. $\sqrt[3]{\dfrac{x^6y^{12n}}{a^{3n}b^{6n}}}$.
9. $\sqrt[3]{}\sqrt{}(729\,a^6x^{12})$.
10. $\sqrt[3]{(27/64)^2}$.
11. $\sqrt[3]{}\sqrt{}(49^3 \times 16^3)$.
12. $\sqrt{(49\,x^{2n}b^6)^3}$.
13. $\sqrt[2n+1]{}(-\,x^{2n+1}y^{4n+2})$.
14. $\sqrt[3]{(8\,a^3x^6y^9)^5}$.

228. Square root by inspection. When a perfect square can be factored by inspection, its square root is found by inspection.

Ex. 1. $$36\,a^4 + b^4 - 12\,a^2b^2 \equiv (6\,a^2 - b^2)^2, \text{ or } (b^2 - 6\,a^2)^2.$$

$$\therefore \sqrt{}(36\,a^4 + b^4 - 12\,a^2b^2) \equiv 6\,a^2 - b^2, \text{ or } b^2 - 6\,a^2.$$

Ex. 2. Find the square root of the first eight expressions in each of the exercises 52 and 53.

229. To show how to find the **square root of any perfect square,** we must show how to *reverse* the process of *squaring* any expression.

E.g., squaring expression (1) we obtain expression (2).

$$x^3 + rx^2 + sx \qquad (1)$$

$$x^6 + 2\,rx^5 + (r^2 + 2\,s)x^4 + 2\,rsx^3 + s^2x^2 \qquad (2)$$

Hence if (2) is taken as a radicand, (1) is its square root.

Now, the *square root* of the *first* term of the *radicand* (2) is the *first* term of the *root* (1).

If we subtract from (2) the *square* of the *first* term of (1), the *first* term of the remainder is $2\,rx^5$. Dividing $2\,rx^5$ by *twice the first* term of the root, $2\,x^3$, we obtain rx^2, the *second* term of the root.

If we subtract from the radicand the *square* of the *sum* of the *first two* terms of the root, $(x^3 + rx^2)^2$, the first term of the remainder is $2\,sx^4$. Dividing $2\,sx^4$ by *twice the first* term of the root, we obtain sx, the *third* term of the root.

This example illustrates the following principles (i) and (ii).

If the terms of a perfect square and its square root are arranged in descending (or ascending) powers of some letter,

(i) *The square root of the first term of the radicand is the first term of the square root.*

(ii) *If the square of the first term in the root, or the square of the sum of its first two or more terms is subtracted from the radicand, and the first term of the remainder is divided by twice the first term of the root, the quotient will be the next term of the root.*

Proof. Let A stand for any number of terms of the square root of any perfect square, and B for the rest; the terms of A and B being arranged in descending (or ascending) powers of the same letter, and every term of A being of a higher (or lower) degree than any term of B.

By § 120, we have the identity

$$A^2 + 2\,AB + B^2 \equiv (A + B)^2. \qquad (1)$$

Let A denote *only* the *first* term of the root; then, since $\sqrt{A^2} = A$, we have (i).

Let A denote the first *one* or *more* terms of the root; then, if we subtract A^2 from the radicand, the remainder is $2\,AB + B^2$. Let a denote the first term of A, and b the first term of B; then, supposing the remainder $2\,AB + B^2$ to be arranged in descending (or ascending) powers of the letter of arrangement, $2\,ab$ will be its first term. Hence, as $2\,ab \div 2\,a \equiv b$, we have (ii).

E.g., by (i), the *first* term of the square root of

$$16\,x^4 - 24\,yx^3 + 25\,y^2x^2 - 12\,y^3x + 4\,y^2 \qquad (3)$$

is $\sqrt{16\,x^4}$, or $4\,x^2$; and by (ii) the second term is $-24\,yx^3 \div 2(4\,x^2)$, or $-3\,yx$.

The radicand (3) less $(4\,x^2 - 3\,yx)^2$ is $16\,y^2x^2 - 12\,y^3x + 4\,y^2$.

Hence, by (ii), the *next* term of the root is $16\,y^2x^2 \div 2(4\,x^2)$, or $2\,y^2$.

The radicand (3) less $(4\,x^2 - 3\,yx + 2\,y^2)^2$ is zero.

Hence, the square root of (3) is $4\,x^2 - 3\,yx + 2\,y^2$.

Instead of finding each square independently, some labor can be saved by using the relation

$$A^2 + (2\,A + b)\,b \equiv (A + b)^2, \tag{2}$$

and thus making use of the previous square. Thus the work in the example above is usually written as below:

$$\begin{array}{lll}
 & 16\,x^4 - 24\,yx^3 + 25\,y^2x^2 - 12\,y^3x + 4\,y^4 & (4\,x^2 - 3\,yx + 2\,y^2. \\
 & \underline{16\,x^4} & \\
8\,x^2 - 3\,yx) & -24\,yx^3 & \\
 & \underline{-24\,yx^3 + 9\,y^2x^2} & \\
8\,x^2 - 6\,yx + 2\,y^2\ ) & 16\,y^2x^2 & \\
 & \underline{16\,y^2x^2 - 12\,y^3x + 4\,y^4} &
\end{array}$$

Subtracting from the radicand the square of the first term of the root, $(4\,x^2)^2$, the first term of the remainder is $-24\,yx^3$.

By (ii), the second term of the root is $-24\,yx^3 \div 2(4\,x^2)$, or $-3\,yx$.

Write $2(4\,x^2) - 3\,yx$ to the left of the first remainder, multiply it by $-3\,yx$, and subtract the product from the first remainder.

Then, by (2), we have subtracted in all

$$(4\,x^2)^2 + (2 \cdot 4\,x^2 - 3\,yx)(-3\,yx), \text{ or } (4\,x^2 - 3\,yx)^2.$$

By (ii), the next term of the root is $16\,y^2x^2 \div 2(4\,x^2)$, or $2\,y^2$.

Write $2(4\,x^2 - 3\,yx) + 2\,y^2$ to the left of the second remainder, multiply it by $2\,y^2$, and subtract the product from the second remainder. Then, by (2), we have subtracted in all

$$(4\,x^2 - 3\,yx)^2 + (8\,x^2 - 6\,yx + 2\,y^2)2\,y^2, \text{ or } (4\,x^2 - 3\,yx + 2\,y^2)^2.$$

As there is no remainder, the required root is $4\,x^2 - 3\,yx + 2\,y^2$.

Observe that we could just as well write radicand (3) in ascending powers of x, or what is the same thing, begin with its last term.

$$16\,x^4 - 24\,yx^3 + 25\,y^2x^2 - 12\,y^3x + 4\,y^4. \tag{3}$$

Thus, by (i), the *last* term of the square root of (3) is $\sqrt{4\,y^2}$, or $2\,y^2$; and, by (ii), the *term before the last* is $-12\,y^3x \div 2\,(2\,y^2)$, or $-3\,yx$, which agrees with the result above.

Ex. Find the square root of $4x^4 - 8x^3 + 4x + 1$. (2)

The *first* term of the root is $2x^2$, and the second term is $-8x^3 \div 4x^2$, or $-2x$.

The *last* term is 1 or -1. If the last term is -1, the term before the last is $4x \div (-2)$, or $-2x$, which is the second term as found above.

But $(2x^2 - 2x - 1)^2 \equiv$ the given expression;

hence $2x^2 - 2x - 1$ is the required root.

If we took $-2x^2$ as the first term of the root, the second term would be $2x$, and the last term 1.

NOTE. In the following exercise the pupil should write out the root at once by (i) and (ii), as in the example above; but he should be drilled also in arranging the work of finding and subtracting the successive squares as on page 237.

Exercise 88.

Find the square root of the following expressions:

1. $x^4 + 2x^3 + 3x^2 + 2x + 1$.
2. $4x^4 - 8x^3 + 4x + 1$.
3. $9x^4 - 36x^3 + 72x + 36$.
4. $4x^4 + 4x^3 - \frac{1}{2}x + \frac{1}{16}$.
5. $x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4$.
6. $x^4 - 2x^3 + \frac{3}{2}x^2 - \frac{1}{2}x + \frac{1}{16}$.
7. $16 - 96x + 216x^2 - 216x^3 + 81x^4$.
8. $1 + 4x + 10x^2 + 12x^3 + 9x^4$.
9. $4x^4 - 4x^3 + 3x^2 - x + \frac{1}{4}$.
10. $1 - xy - \frac{15}{4}x^2y^2 + 2x^3y^3 + 4x^4y^4$.
11. $x^6 - 4x^5 + 6x^4 - 8x^3 + 9x^2 - 4x + 4$.
12. $9x^6 - 12x^5 + 22x^4 + x^2 + 12x + 4$.
13. $x^6 - 22x^4 + 34x^3 + 121x^2 - 374x + 289$.
14. $a^2 - ax + \frac{1}{4}x^2 + 8a - 4x + 16$.

15. $x^8+2x^7+x^6-4x^5-12x^4-8x^3+4x^2+16x+16.$

16. $(1+2x^2)^2-4x(1-x)(1+2x).$

17. $x^4+2x^3(y+z)+x^2(y^2+z^2+4yz)+2xyz(y+z)+y^2z^2.$

18. $x^4-2x+\frac{1}{9}+\frac{29}{3}x^2-6x^3.$

19. $-3a^3+\frac{25}{9}+a^4-5a+\frac{67}{12}a^2.$

20. $\frac{1}{4}x^4+4x^2+\frac{1}{3}ax^2+\frac{1}{9}a^2-2x^3-\frac{4}{3}ax.$

21. $24+\frac{16y^2}{x^2}-\frac{8x}{y}+\frac{x^2}{y^2}-\frac{32y}{x}.$

In the polynomial

$$x^3+x^2+x+1+\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3} \tag{1}$$

each term after the first is obtained by dividing the preceding term by x; hence we regard all the terms in expression (1) as arranged according to the powers of x.

Arranging the given expression according to the powers of x, we have

$$\frac{x^2}{y^2}-\frac{8x}{y}+24-\frac{32y}{x}+\frac{16y^2}{x^2}.$$

22. $\frac{a^4}{4}+\frac{a^3}{x}+\frac{a^2}{x^2}-ax-2+\frac{x^2}{a^2}.$

23. $\frac{9a^2}{x^2}-\frac{6a}{5x}+\frac{101}{25}-\frac{4x}{15a}+\frac{4x^2}{9a^2}.$

24. $4x^4+32x^2+96+\frac{64}{x^4}+\frac{128}{x^2}.$

230. Cube root by inspection. When a perfect cube can be factored by inspection, its cube root is found by inspection.

Ex. 1. $27a^6-54a^5b+36a^4b^2-8a^3b^3 \equiv (3a^2-2ab)^3.$

$\therefore \sqrt[3]{(27a^6-54a^5b+36a^4b^2-8a^3b^3)} \equiv 3a^2-2ab.$

Ex. 2. Find the cube root of the first nine expressions in exercise 60.

231. Cube root of any perfect cube. Let A stand for any number of terms in the cube root of any perfect cube and

B for the rest; the terms of A and B being arranged in descending (or ascending) powers of the same letter, and every term in A being of a higher (or a lower) degree than any term in B.

By § 124 we have the identity

$$A^3 + 3A^2B + 3AB^2 + B^3 \equiv (A+B)^3. \qquad (1)$$

(i) Let A denote the *first* term of the root; then from (1) it follows that *the cube root of the first term of the radicand is the first term of the root.*

(ii) Let A denote the first *one* or *more* terms of the root; then if we subtract A^3 from the radicand the remainder is $3A^2B + 3AB^2 + B^3$. Let a denote the first term of A, and b the first term of B; then supposing the remainder $3A^2B + 3AB^2 + B^3$ to be arranged in descending (or ascending) powers of the letter of arrangement, $3a^2b$ will be its first term. But $3a^2b \div 3a^2 \equiv b$; hence,

If the cube of the first term of the root, or the cube of the sum of its first two or more terms, is subtracted from the radicand, and the first term of the remainder is divided by three times the square of the first term of the root, the quotient will be the next term of the root.

E.g., by (i), the *first* term of the cube root of

$$8x^6 - 36x^5 + 66x^4 - 63x^3 + 33x^2 - 9x + 1$$

is $\sqrt[3]{8x^6}$, or $\mathbf{2x^2}$; and by (ii) the *second* term is $-36x^5 \div 3(2x^2)^2$, or $\mathbf{-3x}$.

The radicand less $(2x^2 - 3x)^3$ is $12x^4 - 36x^3 + 33x^2 - 9x + 1$.
Hence, by (ii), the *next* term of the root is $12x^4 \div 3(2x^2)^2$, or **1**.
The radicand less $(2x^2 - 3x + 1)^3$ is zero.
Hence the cube root is $2x^2 - 3x + 1$.

Instead of finding each cube independently, some labor can be saved by using the relation

$$A^3 + (3A^2 + 3Ab + b^2)b \equiv (A+b)^3,$$

and thus making use of the previous cubes. Thus the work in the example above is usually written as below:

$$\begin{array}{lrl}
 & & \underline{\big|\,2x^2-3x+1} \\
 & & 8x^6-36x^5+66x^4-63x^3+33x^2-9x+1 \\
a^3= & & \underline{8x^6} \\
3a^2=12x^4 & & \big|\,-36x^5 \\
3ab+b^2= & -18x^3+9x^2 & \big|\,\underline{-36x^5+54x^4-27x^3} \\
3A^2=12x^4-36x^3+27x^2 & & \big|\,12x^4-36x^3 \\
3Ab+b^2= & 6x^2-9x+1 & \big|\,\underline{12x^4-36x^3-33x^2-9x+1}
\end{array}$$

By (i), the first term of the root is $\sqrt[3]{8x^6}$, *i.e.*, $a = 2x^2$.

Subtract $(2x^2)^3$; then, by (ii), the second term of the root is $-36x^5 \div 3(2x^2)^2$, *i.e.*, $b = -3x$.

Hence $\qquad 3a^2 + 3ab + b^2 = 12x^4 - 18x^3 + 9x^2.$

Multiply this sum by $-3x$, and subtract the product from the first remainder. Then in all we have subtracted $a^3 + (3a^2 + 3ab + b^2)b$, or $(a+b)^3$; that is, we have subtracted $(2x^2 - 3x)^3$, since $a = 2x^2$ and $b = -3x$.

Let $A =$ *the terms of the root already found* $= 2x^2 - 3x$,

and $\qquad b =$ the *next term* of the root $= 12x^4 \div 3(2x^2)^2 = 1$;

then $\qquad 3A^2 + 3Ab + b^2 = 12x^4 - 36x^3 + 33x^2 - 9x + 1.$

Multiply this sum by 1, and subtract the product from the second remainder. Then in all we have subtracted

$$(A + b)^3, \text{ or } (2x^2 - 3x + 1)^3.$$

As there is no remainder, the required root is $2x^2 - 3x + 1$.

We could just as well write the radicand in ascending powers of x, or, what is the same thing, begin with the last term.

$$8x^6 - 36x^5 + 66x^4 - 63x^3 + 33x^2 - 9x + 1. \qquad (1)$$

Thus, by (i), the *last* term of the cube root of (1) is $\sqrt[3]{1}$, or 1; and, by (ii), the *term before the last* is $-9x \div 3 \cdot 1^2$, or $-3x$, which agrees with the result above.

Ex. Find the cube root of

$$27 + 108x + 90x^2 - 80x^3 - 60x^4 + 48x^5 - 8x^6. \qquad (1)$$

The first term is 3, and the second is $108x \div 3 \cdot 3^2$, or $4x$.

The last term is $-2x^2$, and the term before the last is

$$48x^5 \div 3(-2x^2)^2, \text{ or } 4x.$$

Since $\qquad (3 + 4x - 2x^2)^3 \equiv$ the given expression;

$$3 + 4x - 2x^2 \equiv \text{the required root.}$$

Exercise 89.

Find the cube root of the following expressions:

1. $1 + 3x + 6x^2 + 7x^3 + 6x^4 + 3x^5 + x^6.$

2. $y^6 - 3y^5 + 6y^4 - 7y^3 + 6y^2 - 3y + 1.$

3. $1 - 6x + 21x^2 - 44x^3 + 63x^4 - 54x^5 + 27x^6.$

4. $8a^6 - 36a^5 + 66a^4 - 63a^3 + 33a^2 - 9a + 1.$

5. $8x^6 + 12x^5 - 30x^4 - 35x^3 + 45x^2 + 27x - 27.$

6. $27x^6 - 27x^5 - 18x^4 + 17x^3 + 6x^2 - 3x - 1.$

7. $24x^4y^2 + 96x^2y^4 - 6x^5y + x^6 - 96xy^5 + 64y^6 - 56x^3y^3.$

8. $27x^6 - 54x^5a + 117x^4a^2 - 116x^3a^3 + 117x^2a^4 - 54xa^5 + 27a^6.$

9. $216 + 342x^2 + 171x^4 + 27x^6 - 27x^5 - 109x^3 - 108x.$

10. $x^3 - 9x + \dfrac{27}{x} - \dfrac{27}{x^3}.$

11. $\dfrac{x^6}{y^3} - 6x^4 + 12x^2y^3 - 8y^6.$

12. $\dfrac{x^3}{y^3} + \dfrac{6x^2}{y^2} + \dfrac{9x}{y} - 4 - \dfrac{9y}{x} + \dfrac{6y^2}{x^2} - \dfrac{y^3}{x^3}.$

13. $\dfrac{x^3}{27} - \dfrac{x^2}{3} + 2x - 7 + \dfrac{18}{x} - \dfrac{27}{x^2} + \dfrac{27}{x^3}.$

14. $\dfrac{6b}{a} + \dfrac{6a}{b} - 7 + \dfrac{a^3}{b^3} - \dfrac{3a^2}{b^2} - \dfrac{3b^2}{a^2} + \dfrac{b^3}{a^3}.$

15. $\dfrac{60x^4}{y^4} - \dfrac{80x^3}{y^3} - \dfrac{90x^2}{y^2} + \dfrac{8x^6}{y^6} + \dfrac{108x}{y} - 27 + \dfrac{48x^5}{y^5}.$

232. Higher roots. The *fourth, fifth,* or any other root of a perfect power can be obtained by a method based on one of the following identities:

$$A^4 + 4A^3B + 6A^2B^2 + 4AB^3 + B^4 \equiv (A + B)^4, \qquad (1)$$

$$A^5 + 5A^4B + 10A^3B^2 + 10A^2B^3 + 5AB^4 + B^5 \equiv (A + B)^5. \qquad (2)$$

. . . .

If the terms of a perfect fourth power are *arranged in descending* (or ascending) powers of some letter, from (1) it follows that the first term of the root is the fourth root of the first term of the radicand; that the second term of the root is the second term of the radicand divided by four times the cube of the first term of the root; and that the last term of the root is the fourth root of the last term of the radicand. Similarly for any other higher root.

E.g., the first term of the fourth root of

$$81x^4 + 108x^3 + 54x^2 + 12x + 1 \qquad (1)$$

is $\sqrt[4]{81x^4}$, or $3x$; the second term is $108x^3 \div 4(3x)^3$, or 1, which we know to be the last term of the root.

Since $(3x+1)^4 \equiv$ the radicand (1); $3x+1$ is the fourth root of (1).

Again the first term of the fifth root of

$$32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1 \qquad (2)$$

is $\sqrt[5]{32x^5}$, or $2x$; the second term is $-80x^4 \div 5(2x)^4$, or -1, which we know to be the last term of the root.

Since $(2x-1)^5 \equiv$ the radicand (2); $2x-1$ is the fifth root of (2).

The fourth root can also be obtained by finding the square root of the square root; and the sixth root, by finding the cube root of the square root. Similarly for any other root whose index is not a prime number.

Exercise 90.

By inspection find the fourth root of the expressions:

1. $16a^4 - 96a^3x + 216a^2x^2 - 216ax^3 + 81x^4$.
2. $x^4 - 8x^3a + 24x^2a^2 - 32xa^3 + 16a^4$.
3. $1 + 4a + 4a^7 + 10a^6 + a^8 + 10a^2 + 16a^3 + 16a^5 + 19a^4$.

By inspection find the fifth root of the expressions:

4. $80a^2x^3 - 80ax^4 + 32x^5 - 40a^3x - a^5 + 10a^4x$.
5. $90a^2x^3 - 15ax^4 + x^5 - 270a^3x^2 + 405a^4x - 243a^5$.

By inspection find the sixth root of the expressions:

6. $192x + 64 + 240x^2 + x^6 + 12x^5 + 60x^4 + 160x^3$.

7. $1215a^4 - 1458a^5 - 540a^3 + 135a^2 - 18a + 1 + 729a^6$.

8. $60a^2x^4 - 16a^3x^3 + 64a^6 + x^6 - 12ax^5 + 240a^4x^2 - 192a^5x$.

ROOTS OF DECIMAL NUMBERS.

233. Square roots. $\sqrt{1} = 1$, and $\sqrt{100} = 10$; hence the square root of any number between 1 and 100 lies between 1 and 10; that is, if a number contains one or two integral figures, its square root contains one integral figure.

Again, $\sqrt{100} = 10$, and $\sqrt{10000} = 100$; hence the square root of any number between 100 and 10000 lies between 10 and 100; that is, if a number contains three or four integral figures, its square root contains two figures; and so on.

Hence, in finding the square root of a decimal number, the first step is to divide its integral figures into groups of two figures each, beginning at units' place.

We thus determine the number of integral figures in the root, and indicate the part of the number from which each figure of the root is to be obtained.

The group to the left may contain only one figure.

E.g., in the square root of 5 38 24 there are hundreds, tens, and units; and the hundreds' figure is the square root of the greatest perfect square in 5; that is, the hundreds' figure is 2.

234. From § 233, we have the following principle:

(i) *The first figure in the square root of a decimal number is the square root of the greatest perfect square in the first, or left-hand, group of figures.*

Let A stand for the number denoted by one or more of the first figures of the root, and B stand for the number denoted by the rest; then the root is $A + B$.

Hence the radicand $\equiv A^2 + (2A + B)B$. (1)

Subtract A^2 and then divide by $2A + B$,

$$(\text{radicand} - A^2) \div (2A + B) \equiv B. \quad (2)$$

Let b stand for the number denoted by the first figure in B, *i.e.*, the figure whose order of units is the next lower to the lowest in A; then from (2)

$$(\text{radicand} - A^2) \div 2A > b. \qquad (3)$$

From inequality (3), we have the following principle:

(ii) *If the square of the first part of the root is subtracted from the radicand, and the remainder is divided by twice this part of the root, the quotient will be greater than the next figure of the root.*

E.g., by (i), the *first*, or *hundreds*, figure in the square root of

$$5\,47\,56$$

is 2; since 4 is the greatest perfect square in 5.

The radicand less $(200)^2$ is 14756.

Hence, by (ii), the *tens* figure of the root cannot exceed

$$14756 \div 2(200), \text{ or } 3 \text{ tens}.$$

The radicand less $(230)^2$ is 1856; hence the root is greater than 230, and 3 is the tens figure of the root.

By (ii), the *units* figure of the root cannot exceed

$$1856 \div 2(230), \text{ or } 4.$$

The radicand less $(234)^2$ is zero.

Hence the square root of 54756 is 234.

Instead of finding each square independently, much labor can be saved by using the relation (§ 229)

$$A^2 + (2A + b)b \equiv (A + b)^2,$$

and thus making use of the previous square.

The work in the example above is usually written as below:

$$\begin{array}{rr}
 & 5\,47\,56(234 \\
A^2 = (200)^2 = & \underline{4\,00\,00} \\
2A + b = 2(200) + 30 = & 430)1\,47\,56 \\
(2A + b)b = 430 \times 30 = & \underline{1\,29\,00} \\
2A + b = 2(230) + \ \ 4 = & 464)18\,56 \\
(2A + b)b = 464 \times \ \ 4 = & \underline{18\,56}
\end{array}$$

At first $A = 200$. Subtracting A^2, or 200^2, from the radicand, and dividing the remainder, 14756, by $2A$, or $2(200)$, we find that the tens figure of the root cannot exceed 3.

Multiply $2A + b$, or 430, by b, or 30, and subtract the product; then in all we have subtracted $A^2 + (2A + b)b$, or $(A + b)^2$; that is, 230^2.

Now let $A = 230$, *the part of the root already found,*

and $b =$ the *next* figure of the root.

Dividing the remainder 1856 by $2A$, or 460, we find that the units figure of the root cannot exceed 4.

Multiply $2A + b$, or 464, by b, or 4, and subtract the product; then in all we have subtracted $A^2 + (2A + b)b$, or $(A + b)^2$; that is, $(234)^2$.

Omitting the ciphers and explanation, and in each remainder writing the next group of figures only, the work will stand as below:

$$\begin{array}{r|l}
 & 5\,47\,56(234 \\
 & 4 \\
\hline
43) & 1\,47 \\
 & 1\,29 \\
\hline
464) & 18\,56 \\
 & 18\,56 \\
\hline
\end{array}$$

235. If a number has decimal places, its *square* will have *twice* as many. *E.g.*, $0.8^2 = 0.64$; $0.25^2 = 0.0625$.

Hence to determine how many decimal figures there will be in the square root of a number, we divide its decimal figures into groups of two figures each, beginning at the decimal point. If the group to the right does not contain two figures, a cipher must be annexed.

Ex. Find the square root of 5727.2976.

Formula, $A^2 + (2A + b)b \equiv (A + b)^2$.

$$\begin{array}{r|l}
 & 75\,27\,.29\,76(86.76 \\
 & 64 \\
\hline
166) & 11\,27 \\
 & 9\,96 \\
\hline
1727) & 1\,31\,29 \\
 & 1\,20\,89 \\
\hline
17346) & 10\,40\,76 \\
 & 10\,40\,76 \\
\hline
\end{array}$$

Here at first $A = 80$, $b = 6$; next $A = 86$, $b = 0.7$, next $A = 86.7$, $b = 0.06$.

Exercise 91.

Find the square root of the numbers:

1. 2916.	9. 29376400.	17. 0.0022448644.
2. 2601.	10. 52.2729.	18. 0.68112009.
3. 17956.	11. 53.7289.	19. 25/49.
4. 33489.	12. 883.2784.	20. 64/81.
5. 119025.	13. 1.97262025.	21. 121/36.
6. 15129.	14. 3080.25.	22. 144/49.
7. 103041.	15. 41.2164.	23. 169/196.
8. 835396.	16. 384524.01.	24. 225/289.

236. Cube root. Since $\sqrt[3]{1} = 1$, and $\sqrt[3]{1000} = 10$, it follows that the cube root of any number between 1 and 1000 lies between 1 and 10; that is, if a number contains one, two, or three integral figures, its cube root contains one integral figure. Again, $\sqrt[3]{1000} = 10$, and $\sqrt[3]{1000000} = 100$; hence, if a number contains four, five, or six integral figures, its cube root contains two integral figures; and so on.

Hence, to determine how many integral figures there are in the cube root of a number, we divide its integral figures into groups of three figures each, beginning at units' place. The last group to the left may contain only one or two figures.

When the figures of a number have been divided into groups of three figures each, from what precedes it follows that,

(i) *The first figure in the cube root of a decimal number is the cube root of the greatest cube in the first, or left-hand, group of figures.*

Using a notation analogous to that in § 234, we have

$$\text{radicand} - A^3 = (3\,A^2 + 3\,AB + B^2)\,B.$$

$$\because\ (\text{radicand} - A^3) \div (3\,A^2 + 3\,AB + B^2) = B.$$

$$\therefore\ (\text{radicand} - A^3) \div 3\,A^2 > b. \qquad (1)$$

From inequality (1) it follows that,

(ii) *If the cube of the first part of the root is subtracted from the radicand and the remainder is divided by three times the square of this part of the root, the quotient will be greater than the next figure of the root.*

E.g., by (i), the *first* or *tens'* figure in the cube root of

$$614\,125$$

is 8, since 8^3, or 512, is the greatest perfect cube in 614.

The radicand less $(80)^3$ is 102125.

Hence, by (ii), the units' figure of the root cannot exceed

$$102125 \div 3(80)^2, \text{ or } 5.$$

The radicand less $(85)^3$ is zero.

Hence, the required root is 85.

Instead of finding each cube independently, much labor can be saved by using the relation

$$A^3 + (3\,A^2 + 3\,A\,b + b^2)\,b \equiv (A + b)^3,$$

and thus making use of the previous cubes.

Thus, the work in the example above is usually written as below, without the explanations to the left:

$$\begin{array}{rrrr|l}
 & & & & 614\,125(85 \\
A^3 = & & & & \underline{512\,000} \\
3\,A^2 = & 3(80)^2 & = & 19200) & 102\,125 \\
3\,Ab = & 3 \cdot 80 \cdot 5 & = & 1200 & \\
b^2 = & 5^2 & = & \underline{25} & \\
 & & & 20425 & 102\,125
\end{array}$$

237. If a number has decimal places, its *cube* will have *three* times as many. Thus $0.2^3 = 0.008$; $0.12^3 = 0.001728$. Hence, to determine how many decimal figures there will be in the *cube root* of a number, we divide its decimal figures into groups of three figures each, beginning at the decimal point.

If the group to the right does not contain three figures, ciphers must be annexed.

Ex. Find the cube root of 129554.216.

Formula, $A^3 + (3A^2 + 3Ab + b^2)b \equiv (A+b)^3$.

$$
\begin{array}{r|l}
 & 129\,554\,.216(50.6 \\
 & 125 \\ \hline
750000 & 4\,554\,216 \\
9000 & \\
36 & \\ \hline
759036 & 4\,554\,216 \\ \hline
\end{array}
$$

Here at first $A = 50$, $b = 0$; next $A = 50.0$, $b = 0.6$.

Exercise 92.

Find the cube root of the numbers:

1. 74088.
2. 15625.
3. 32768.
4. 110592.
5. 262144.
6. 1481544.
7. 103.823.
8. 884.736.
9. 1953125.
10. 7077888.
11. 2.803221.
12. 12.812904.
13. 56.623104.
14. 264.609288.
15. 1076890625.
16. 8/27.
17. 64/125.
18. 343/1728.

INCOMMENSURABLE ROOTS, OR IRRATIONAL NUMBERS.

238. The nth power of a whole number is evidently a whole number which is a perfect nth power; and the nth power of a fraction (whose numerator and denominator are prime to each other) is a fraction whose numerator and denominator are perfect nth powers prime to each other.

Hence, it follows that

(i) *The* n*th root of a whole number which is not the* n*th power of another whole number is not a commensurable number.*

(ii) *The* n*th root of a fraction whose numerator and denominator (prime to each other) are not the* n*th powers of whole numbers, is not a commensurable number.*

E.g., as 2 is not the square of any whole number, $\sqrt{2}$ is not a commensurable number, and therefore is not as yet included in our number system. The same is true of $\sqrt{3}$, $\sqrt{5}$, $\sqrt[4]{7}\cdots$.

Again, as the terms of the fraction $2/3$ are prime to each other and are not the squares of whole numbers, $\sqrt{(2/3)}$ is not a commensurable number.

239. To enlarge our number concept so as to give meaning to such expressions as $\sqrt{2}$, $\sqrt[3]{5}$, etc., we assume the identity

$$(\sqrt[n]{u})^n \equiv u$$

to hold when the radicand u is not a perfect nth power.

E.g., $\sqrt{2}$ is the number whose square is 2, *i.e.* $(\sqrt{2})^2 = 2$.

Again, $\sqrt[3]{5}$ is the number whose cube is 5, *i.e.* $(\sqrt[3]{5})^3 = 5$.

240. The nth root of a number which is not a perfect nth power is called an **incommensurable root** or an **irrational number**; as, $\sqrt{2}$, $\sqrt[4]{3}$.

241. An irrational number, or any other number which is not a whole or a fractional number, is called an **incommensurable number**; as $\sqrt{3}$, $\sqrt[3]{5}$, or the ratio of the circumference of a circle to its diameter.

242. Approximate values of incommensurable roots.

If to 2 we add ciphers and apply the method of finding the square root, we obtain the result below:

$$\begin{array}{rl}
2.00000000\,)\underline{1.4142\cdots} & \\
\underline{1} & \\
24)100 & \text{1st remainder}\\
\underline{90} & \\
281)400 & \text{2d remainder}\\
\underline{281} & \\
2824)11900 & \text{3d remainder}\\
\underline{11296} & \\
28282)60400 & \text{4th remainder}\\
\underline{56564} & \\
0.00003836 & \text{5th remainder}
\end{array}$$

Each remainder in the above process is the difference between 2 and the square of the corresponding part of the root.

This remainder decreases rapidly as we increase the number of figures in the root; hence the square of the root found approaches nearer and continually nearer 2; and therefore the root itself approaches nearer and continually nearer $\sqrt{2}$.

By continuing the operation indefinitely we obtain a commensurable number which approaches indefinitely near and continually nearer $\sqrt{2}$, but which, by § 238, can never reach $\sqrt{2}$. This increasing commensurable number is said *to approach the incommensurable root* $\sqrt{2}$ *as its limit.*

In like manner we can find a commensurable number which shall differ from any incommensurable root by as little as we please.

Exercise 93.

Obtain to three places of decimals the value of the roots:

1. $\sqrt{3}$.	4. $\sqrt{7}$.	7. $\sqrt{0.3}$.	10. $\sqrt{0.004}$.	13. $\sqrt[3]{2}$.
2. $\sqrt{5}$.	5. $\sqrt{11}$.	8. $\sqrt{0.5}$.	11. $\sqrt{0.005}$.	14. $\sqrt[3]{4}$.
3. $\sqrt{6}$.	6. $\sqrt{13}$.	9. $\sqrt{0.03}$.	12. $\sqrt{2.5}$.	15. $\sqrt[3]{2.5}$.

243. The quality-unit $+1$ or -1 multiplied by an arithmetic incommensurable number is a *positive* or a *negative incommensurable number* ; as, $+\sqrt{2}$, $-\sqrt{3}$, $-\sqrt[3]{5}$.

244. The fundamental laws which have been proved for commensurable numbers hold also for incommensurable numbers. The proof of these laws for incommensurable numbers will be found in the chapter on the theory of limits.

245. An **irrational expression** is one which involves the nth root of an expression which is not a perfect nth power; as, $\sqrt[3]{x^2}$, $\sqrt{(a+b)}$. Any *irrational numeral* expression denotes an irrational number. But, just as a fractional literal

expression denotes both integral and fractional numbers, so an irrational literal expression denotes both rational and irrational numbers.

E.g., the irrational literal expression $\sqrt{a}$ denotes a commensurable or rational number, when $a = 1, 4, 9, 1/4, 4/9, \cdots$ and an incommensurable or irrational number when $a = 2, 3, 5, \cdots$.

Observe that *commensurable* and *incommensurable* apply to *numbers only*; while *rational* and *irrational* apply to either numbers or expressions.

Exercise 94.

1. Is the number $\sqrt{4}$ commensurable or incommensurable? $\sqrt{6}$? $\sqrt{9}$? $\sqrt{12}$? $\sqrt{14}$? $\sqrt{(4/9)}$? $\sqrt[3]{(8/27)}$? $\sqrt[3]{3}$? $\sqrt[3]{216}$?

2. Is the expression $\sqrt{a^4}$ rational or irrational? $\sqrt{x}$? $\sqrt[3]{a^6}$? $\sqrt{(a+b)}$? $\sqrt{(a+x)^4}$? $\sqrt{(a/x)}$?

3. Give ten sets of values of a and x for which $\sqrt{(a/x)}$ denotes a commensurable, or rational, number.

CHAPTER XVII

SURDS

246. A **surd number** is an irrational number in which the radicand is a rational number; in other words, it is an incommensurable root of a commensurable number.

E.g., $\sqrt{5}$, $\sqrt[3]{7}$, $\sqrt[5]{(2/3)}$ are surd numbers; so also is $\sqrt{a}$ when a denotes a commensurable number which is not a perfect square.

The incommensurable root $\sqrt{(3+\sqrt{2})}$ is not a surd number, since the radicand $3+\sqrt{2}$ is not a commensurable number.

247. A **surd expression** is an irrational expression in which each radicand is a rational expression; as $\sqrt{a}$, $\sqrt{5/6}$, $\sqrt{x}+\sqrt{2}$.

248. Surds of different orders. A *surd of the second order*, or a *quadratic surd*, is a surd with the index 2; as $\sqrt{5}$, $\sqrt{a}$.

A *surd of the nth order* is a surd with the index n; as $\sqrt[n]{a}$.

Observe that $\sqrt[3]{\sqrt{5}}$ is a surd of the 6th order.

249. A rational number or expression can be written in the *form* of a surd of any order.

E.g., $3 = \sqrt{9},\ \sqrt[3]{27},\ \sqrt[4]{81}, \text{ or } \sqrt[5]{243}$

and $a \equiv \sqrt{a^2},\ \sqrt[3]{a^3},\ \sqrt[4]{a^4}, \text{ or } \sqrt[5]{a^5}.$

250. A surd expression is in its *simplest form* when each radicand is *integral*, its numeral factor being as small as possible, and its literal factor of as low degree as possible.

E.g., the simplest form of the surd $\sqrt{8}$ is $2\sqrt{2}$.

The simplest form of the surd $\sqrt[3]{(16\,x^4y^5)}$ is $2\,xy\sqrt[3]{(2\,x^2y)}$.

251. Reduction of surds to their simplest form. The cases which most frequently occur are the three following:

I. *Radicand integral.* Resolve the radicand into two factors, one of which is a perfect power of a degree equal to the order of the surd, and apply the law, $\sqrt[n]{(ab)} \equiv \sqrt[n]{a} \cdot \sqrt[n]{b}$.

Ex. 1. $\sqrt[3]{135} = \sqrt[3]{3^3 \cdot 5} = 3\sqrt[3]{5}.$

Ex. 2. $7\sqrt{50\,x^4y^3} = 7\sqrt{(5\,x^2y)^2 \cdot 2\,y}$

$= 7 \times 5\,x^2y\sqrt{2\,y} = 35\,x^2y\sqrt{2\,y}.$

Ex. 3. $5\sqrt[3]{128\,x^5y^4} = 5\sqrt[3]{(4\,xy)^3 \cdot 2\,x^2y}$

$= 5 \times 4\,xy\sqrt[3]{2\,x^2y} = 20\,xy\sqrt[3]{2\,x^2y}.$

II. *Radicand fractional.* Multiply both the numerator and denominator by such a number as will make the denominator a perfect power of a degree equal to the order of the surd, and apply the law, $\sqrt[n]{(a/b)} \equiv \sqrt[n]{a}/\sqrt[n]{b}$.

Ex. 1. $\sqrt[3]{\frac{5}{49}} = \sqrt[3]{\frac{35}{7^3}} = \frac{\sqrt[3]{35}}{7}.$

Ex. 2. $\sqrt[3]{\frac{3\,x}{5\,ab^5}} \equiv \sqrt[3]{\frac{75\,a^2bx}{(5\,ab^2)^3}} = \frac{\sqrt[3]{75\,a^2bx}}{5\,ab^2}.$

III. *Factor common to exponent of radicand and index of root.*

In this case we apply the following principle:

The index of the root and the exponent of the radicand can be multiplied or divided by the same number.

That is, $\sqrt[nr]{a^{mr}} \equiv \sqrt[n]{\sqrt[r]{a^{mr}}} \equiv \sqrt[n]{a^m}.$ §§ 223, 227

Ex. 1. $\sqrt[6]{(27\,a^3x^3)} \equiv \sqrt[6]{(3\,ax)^3} \equiv \sqrt{(3\,ax)}.$

Ex. 2. $\sqrt[8]{(16\,a^4x^{12})} \equiv \sqrt[8]{x^8 \cdot (2\,ax)^4}$

$\equiv \sqrt[8]{x^8} \cdot \sqrt[8]{(2\,ax)^4} \equiv x\sqrt{(2\,ax)}.$

Exercise 95.

Reduce each of the following surds to its simplest form:

1. $\sqrt{147}$.
2. $\sqrt{288}$.
3. $3\sqrt{150}$.
4. $2\sqrt{720}$.
5. $\sqrt[3]{256}$.
6. $\sqrt[3]{432}$.
7. $5\sqrt{245}$.
8. $\sqrt[3]{1029}$.
9. $\sqrt[4]{3125}$.
10. $\sqrt[3]{-2187}$.
11. $\sqrt{27\,a^3b^5}$.
12. $\sqrt[3]{-108\,x^4y^3}$.
13. $\sqrt[n]{x^{3n}y^{2n+5}}$.
14. $\sqrt[p]{x^{a+p}y^{2p}}$.
15. $\sqrt{a^3+2\,a^2b+ab^2}$.
16. $\frac{1}{2}\sqrt{\frac{1}{3}}$.
17. $\frac{2}{3}\sqrt{\frac{4}{5}}$.
18. $\frac{4}{11}\sqrt{\frac{77}{2}}$.
19. $3\sqrt[3]{\frac{2}{3}}$.
20. $\dfrac{3\,xy}{z}\sqrt{\dfrac{5\,z^2}{9\,x^2y}}$.
21. $\dfrac{3\,x}{2\,a}\sqrt[3]{\dfrac{27\,a^4}{x^4}}$.
22. $\dfrac{2\,b}{a}\sqrt[4]{\dfrac{a^4}{8\,b^3}}$.
23. $a\sqrt[5]{\dfrac{b^2}{a^3}}$.
24. $\dfrac{a}{x}\sqrt[n]{\dfrac{x^{n+1}}{a^{n-2}}}$.
25. $\dfrac{a}{b^n}\sqrt{\dfrac{b^{2n+1}}{a^5}}$.
26. $(a+b)\sqrt{\dfrac{a-b}{a+b}}$.
27. $\dfrac{c}{b}\sqrt{\dfrac{a^2-x^2}{x^3c^3}}$.
28. $\sqrt[4]{25}$.
29. $\sqrt[4]{(8/x^2)}$.
30. $\sqrt[6]{(x^3)}$.
31. $\sqrt[14]{(9\,x^6)}$.
32. $\sqrt[4]{(9/36)}$.
33. $\sqrt[6]{(x^2/y^2)}$.
34. $\sqrt[10]{(32/x^{15})}$.
35. $\sqrt[mx]{(2^x/a^{nx})}$.
36. $\sqrt[ms]{(3^{rs}/x^{ns})}$.
37. $\sqrt{(b-c)(b^3-c^3)}$.
38. $\sqrt[3]{(-54\,x^{4n+11}y^{14})}$.
39. $\sqrt[12]{(64\,a^8x^{10})}$.

252. Surds which are rational multiples of the same monomial surd are said to be **like, or similar, surds.**

E.g., $2\sqrt{3}$ and $\sqrt{3}/7$ are *like* surds, so also are $b\sqrt{a}$ and $\sqrt{4\,a}$, or $2\sqrt{a}$.

253. To *add or subtract* surds, we reduce them to their simplest form, and unite those that are similar.

Ex. 1. $\sqrt[3]{135}+\sqrt[3]{40}=3\sqrt[3]{5}+2\sqrt[3]{5}=5\sqrt[3]{5}$.

Ex. 2. $4\sqrt{128}+4\sqrt{75}-5\sqrt{162}=32\sqrt{2}+20\sqrt{3}-45\sqrt{2}$
$=20\sqrt{3}-13\sqrt{2}$.

Exercise 96.

Simplify each of the following surd expressions:

1. $\sqrt{27}+\sqrt{48}$.
2. $2\sqrt{180}-\sqrt{405}$.
3. $2\sqrt{28}-\sqrt{63}$.
4. $5\sqrt{208}-3\sqrt{325}$.
5. $\sqrt{512}-\sqrt{50}-\sqrt{98}$.
6. $3\sqrt{12}-\sqrt{27}+2\sqrt{75}$.
7. $\sqrt{44}-5\sqrt{176}+2\sqrt{99}$.
8. $2\sqrt{363}-5\sqrt{243}+\sqrt{192}$.
9. $2\sqrt[3]{189}+3\sqrt[3]{875}-7\sqrt[3]{56}$.
10. $\sqrt[3]{81}-7\sqrt[3]{192}+4\sqrt[3]{648}$.
11. $\sqrt{252}-\sqrt{294}-48\sqrt{\frac{1}{6}}$.
12. $4\sqrt{63}+5\sqrt{7}-8\sqrt{28}$.
13. $\sqrt[3]{a^2}+\frac{1}{2}\sqrt[3]{a^2}-3\sqrt[3]{27\,a^2}$.
14. $\sqrt[3]{54}+\sqrt{\frac{1}{2}}-\frac{3}{4}\sqrt{\frac{2}{9}}$.
15. $\sqrt[3]{27\,c^4}-\sqrt[3]{8\,c^4}+\sqrt[3]{125\,c}$.
16. $\sqrt[5]{a^5b}-\sqrt[5]{b^6}+\sqrt[5]{32\,b}$.
17. $\sqrt{a^4x}+\sqrt{b^4x}-\sqrt{4\,a^2b^2x}$.
18. $3\sqrt{147}-\frac{7}{3}\sqrt{\frac{1}{3}}-\sqrt{\frac{1}{27}}$.
19. $3\sqrt{\frac{4}{5}}+3\sqrt{\frac{9}{5}}-\sqrt[4]{\frac{25}{81}}$.
20. $\frac{1}{18}\sqrt[3]{72}-\frac{1}{3}\sqrt[3]{\frac{1}{3}}+6\sqrt[3]{21\frac{1}{3}}$.
21. $\sqrt[4]{(9\,x^2y^2)}+\sqrt{(27\,x^3y)}+5\sqrt[4]{(729\,x^6y^2)}$.
22. $2\sqrt[3]{(3\,a^2b)}-\sqrt[6]{(9\,a^4b^2)}+\sqrt[3]{(125\,a^4b)}$.
23. $\sqrt{(4\,a^3+4\,a^2b)}+\sqrt{(9\,ab^2+9\,b^3)}$.
24. $\sqrt{x^3-x^2y}-\sqrt{xy^2-y^3}-\sqrt{(x+y)(x^2-y^2)}$.

254. Surds of different orders can be reduced to identical surds of the same order. This order can be *any* common multiple of each of the given orders, but it is usually most convenient to choose the *least* common multiple.

Ex. 1. Reduce $\sqrt[3]{a^2}$, $\sqrt[4]{b^3}$, $\sqrt[6]{c^5}$ to identical surds of the same order.

The L. C. M. of the indices 3, 4, and 6 is 12. By III. of § 251,

$$\sqrt[3]{a^2}\equiv\sqrt[12]{a^8};\ \sqrt[4]{b^3}\equiv\sqrt[12]{b^9};\ \sqrt[6]{c^5}\equiv\sqrt[12]{c^{10}}.$$

Ex. 2. Which is the greater $\sqrt[3]{6}$ or $\sqrt[4]{10}$?

Reducing these surds to the same order, we have

$$\sqrt[3]{6} = \sqrt[12]{6^4} = \sqrt[12]{1296}, \tag{1}$$

and

$$\sqrt[4]{10} = \sqrt[12]{10^3} = \sqrt[12]{1000}. \tag{2}$$

From their values in (1) and (2), it follows that $\sqrt[3]{6} > \sqrt[4]{10}$.

255. The *product* of two or more surds is found by applying the law

$$\sqrt[n]{a}\sqrt[n]{b} \equiv \sqrt[n]{(ab)}. \qquad \S\ 222$$

Ex. 1. $\sqrt{7} \times \sqrt{28} = \sqrt{7} \times 2\sqrt{7} = 2 \times 7 = 14.$

Ex. 2. $2\sqrt{14} \times \sqrt{21} = 2\sqrt{(14 \times 21)} = 2\sqrt{(7^2 \times 6)} = 14\sqrt{6}.$

When the surds are of different orders, they should be reduced to the same order.

Ex. 3. $\sqrt{3} \times \sqrt[3]{2} = \sqrt[6]{3^3} \times \sqrt[6]{2^2} = \sqrt[6]{(3^3 \times 2^2)} = \sqrt[6]{108}.$

Conversely to § 251, the coefficient of a surd can be brought under the radical sign by reducing it to the form of a surd of the same order.

Ex. 4. $5\sqrt{3} = \sqrt{25} \times \sqrt{3} = \sqrt{75}.$

Ex. 5. $x\sqrt[5]{x^3} \equiv \sqrt[5]{x^5} \times \sqrt[5]{x^3} \equiv \sqrt[5]{x^8}.$

Ex. 6. Multiply $2\sqrt{3} + 3\sqrt{2}$ by $4\sqrt{3} - 5\sqrt{2}$.

The work can be arranged as below:

$$\begin{array}{l} 2\sqrt{3} + 3\sqrt{2} \\ 4\sqrt{3} - 5\sqrt{2} \\ \hline \quad 24 + 2\sqrt{6} - 30 = 2\sqrt{6} - 6. \end{array}$$

256. In finding powers of monomial surds we often make use of the law, $(\sqrt[s]{a})^r \equiv \sqrt[s]{a^r}$ (§ 226).

Ex. 1. $(3\sqrt[3]{ax})^2 \equiv 3^2(\sqrt[3]{ax})^2$ § 119

$$\equiv 9\sqrt[3]{(ax)^2} \equiv 9\sqrt[3]{(a^2x^2)}.$$

Ex. 2. $(2\sqrt{x})^3 \equiv 2^3(\sqrt{x})^3 \equiv 8x\sqrt{x}.$

When applicable, the identities in Chap. IX. should be used in finding the products of polynomial surds.

Ex. 3. $(\sqrt{3}-\sqrt{5})^2 = (\sqrt{3})^2 - 2\sqrt{3}\cdot\sqrt{5} + (\sqrt{5})^2$
$= 3 - 2\sqrt{15} + 5 = 8 - 2\sqrt{15}.$

257. Two binomial quadratic surds which differ only in the quality of a surd term are called **conjugate surds.**

E.g., $3+\sqrt{2}$ and $3-\sqrt{2}$ are conjugate surds; so also are $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$, or $\sqrt{a}+\sqrt{b}$ and $-\sqrt{a}+\sqrt{b}$.

The product of two conjugate surds is rational.

E.g., $(3+\sqrt{2})(3-\sqrt{2}) = 3^2 - (\sqrt{2})^2 = 7.$

$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) \equiv (\sqrt{a})^2 - (\sqrt{b})^2 \equiv a - b.$

Exercise 97.

Reduce to surds of the same order:

1. $\sqrt{3}$, $\sqrt[4]{7}$.
2. $\sqrt{(1/2)}$, $\sqrt[3]{(2/3)}$.
3. $\sqrt[3]{2}$, $\sqrt[4]{3}$.
4. $\sqrt[4]{8}$, $\sqrt{3}$, $\sqrt[8]{6}$.
5. $\sqrt{5}$, $\sqrt[3]{11}$, $\sqrt[6]{13}$.
6. $\sqrt{a}$, $\sqrt[3]{a}$, $\sqrt[4]{a}$.
7. 2, $\sqrt[3]{3}$, $\sqrt{4}$.
8. $\sqrt[4]{3}$, 2, $\sqrt[8]{7}$.
9. $\sqrt[m]{a^2}$, b^2, $\sqrt[n]{c}$.

Bring the coefficient under the radical sign:

10. $11\sqrt{2}$.
11. $14\sqrt{5}$.
12. $6\sqrt[3]{4}$.
13. $5\sqrt[3]{6}$.
14. $\dfrac{4}{11}\sqrt{\dfrac{77}{8}}$.
15. $\dfrac{3\,ab}{2\,c}\sqrt{\dfrac{20\,c^2}{9\,a^2b}}$.
16. $\dfrac{2\,a}{3\,x}\sqrt[3]{\dfrac{27\,x^4}{a^2}}$.

Which is the greater:

17. $\sqrt{5}$ or $\sqrt[3]{10}$?
18. $\sqrt{3}$ or $\sqrt[3]{5\frac{1}{5}}$?
19. $5\sqrt{2}$ or $\sqrt[3]{344}$?
20. $\sqrt[3]{5}$ or $\sqrt[4]{10}$?
21. $\sqrt[3]{a^2}$ or $\sqrt{a}$, when $a<1$? when $a>1$?

Obtain the simplest form for each of the following products:

22. $2\sqrt{15} \times 3\sqrt{5}$.

23. $8\sqrt{12} \times 3\sqrt{24}$.

24. $\sqrt{12} \times \sqrt{27} \times \sqrt{75}$.

25. $\sqrt[3]{16} \times \sqrt[3]{6} \times \sqrt[3]{9}$.

26. $\sqrt[3]{12} + \sqrt[3]{75} \times \sqrt[3]{30}$.

27. $\sqrt[4]{6} \times \sqrt[4]{12} \times \sqrt[4]{18}$.

28. $\sqrt[3]{x+2} \times \sqrt[3]{x-2}$.

29. $\sqrt[3]{168} \times \sqrt[3]{147}$.

30. $5\sqrt[3]{128} \times 2\sqrt[3]{432}$.

31. $\sqrt{10} \times \sqrt[3]{200}$.

32. $\sqrt[3]{4} \times \sqrt{8}$.

33. $(\sqrt{6}-\sqrt{3})(\sqrt{6}+\sqrt{3})$.

34. $(\sqrt{6}-\sqrt{7})(\sqrt{6}+\sqrt{\;})^{-}$.

35. $(\sqrt{c}-\sqrt{x})(\sqrt{c}+\sqrt{x})$.

36. $(-\sqrt{c}-\sqrt{a})(-\sqrt{c}+\sqrt{a})$.

37. $(-\sqrt{xy}+\sqrt{r})(\sqrt{xy}+\sqrt{r})$.

Find each of the following powers:

38. $(\sqrt{2})^3$.

39. $(2\sqrt{3})^4$.

40. $(\sqrt{x})^5$.

41. $(\sqrt[3]{a})^5$.

42. $(\sqrt[3]{ab})^4$.

43. $(\sqrt[3]{a^2})^5$.

44. $(\sqrt[4]{a^3})^6$.

45. $(\sqrt[5]{b^3})^{11}$.

46. $(\sqrt[3]{a^2x^5})^8$.

47. $(2\sqrt{a^3bx})^4$.

48. $(3\sqrt[3]{a^2b^4x})^5$.

49. $(2\sqrt[3]{x^2-y^2})^6$.

50. $(\sqrt{3}-\sqrt{5})^2$.

51. $(4-2\sqrt{3})^2$.

52. $(\sqrt{5}+2\sqrt{3})^2$.

53. $(\sqrt{3}-\sqrt{2})^3$.

54. $(\sqrt[4]{2}-\sqrt[4]{4})^2$.

55. $(\sqrt{6}-\sqrt[3]{2})^3$.

56. $(\sqrt{2}+\sqrt{3}+\sqrt{5})^2$.

57. $(1+\sqrt{2}+\sqrt{3})^3$.

Find each of the following products, and simplify:

58. $(2\sqrt{5}+3\sqrt{3})(3\sqrt{5}-4\sqrt{3})$.

59. $(\sqrt{2}+\sqrt{3}+\sqrt{6})(2\sqrt{2}+3\sqrt{3}+\sqrt{6})$.

60. $(5+\sqrt[3]{4})(\sqrt{3}+\sqrt{2})$.

61. $(2\sqrt{3}+\sqrt[3]{2})(2\sqrt{3}-\sqrt[3]{4})$.

62. $(8-3\sqrt{7})(8+3\sqrt{7})$.

63. $(1+\sqrt{2}-\sqrt{3})^2$.

64. $(\sqrt{2}+\sqrt{3}-\sqrt{5})(\sqrt{2}+\sqrt{3}+\sqrt{5})$.

258. Division of surds.

Suppose it is required to compute the value of $\sqrt{5}/\sqrt{7}$. We might find $\sqrt{5}$, which is $2.236\cdots$; then find $\sqrt{7}$, which is $2.645\cdots$; and finally divide $2.236\cdots$ by $2.645\cdots$.

Of these three long operations two will be avoided if we first multiply both dividend and divisor by $\sqrt{7}$, as below:

$$\begin{aligned}\sqrt{5}/\sqrt{7} &= \sqrt{35}/7 \\ &= 5.916\cdots/7 = 0.845\cdots.\end{aligned}$$

Observe that the new divisor is a rational number.

This example illustrates the following principle:

The quotient of one surd divided by another is put in the simplest form for computation by multiplying both dividend and divisor by such a factor as will render the *divisor rational.*

This process is called *rationalizing the divisor*, or *rationalizing the denominator.*

The factor by which we multiply the divisor to obtain a rational divisor is called the **rationalizing factor**; as, $\sqrt{7}$ above.

The cases which most frequently occur are the three following:

I. When the divisor is a monomial surd; as, $\sqrt[n]{x^m}$.

Ex. 1. $\dfrac{2}{3\sqrt{5}}=\dfrac{2\times\sqrt{5}}{3\sqrt{5}\times\sqrt{5}}=\dfrac{2}{15}\sqrt{5}$.

Ex. 2. $\dfrac{b}{c\sqrt[5]{a^2}}\equiv\dfrac{b\times\sqrt[5]{a^3}}{c\sqrt[5]{a^2}\times\sqrt[5]{a^3}}\equiv\dfrac{b}{ca}\sqrt[5]{a^3}$.

Here the *rationalizing factor* is $\sqrt[5]{a^3}$.

The simplest rationalizing factor of $\sqrt[n]{x^m}$ is evidently $\sqrt[n]{a^{n-m}}$.

II. When the divisor is a binomial quadratic surd, the simplest rationalizing factor is the conjugate of the divisor.

Ex. 1. $\dfrac{5+\sqrt{7}}{3-\sqrt{7}}=\dfrac{(5+\sqrt{7})(3+\sqrt{7})}{(3-\sqrt{7})(3+\sqrt{7})}=\dfrac{22+8\sqrt{7}}{9-7}=11+4\sqrt{7}.$

Ex. 2. $\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}\equiv\dfrac{(\sqrt{a}+\sqrt{b})(\sqrt{a}+\sqrt{b})}{(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})}\equiv\dfrac{a+2\sqrt{ab}+b}{a-b}.$

III. When the divisor is of the form $(\sqrt{a}+\sqrt{b})+\sqrt{c}$, first multiply by the expression $(\sqrt{a}+\sqrt{b})-\sqrt{c}$.

The divisor thus becomes

$$(\sqrt{a}+\sqrt{b})^2-(\sqrt{c})^2, \text{ or } (a+b-c)+2\sqrt{(ab)}. \qquad (1)$$

Next we multiply by the conjugate surd

$$(a+b-c)-2\sqrt{(ab)}.$$

The divisor thus becomes the rational expression

$$(a+b-c)^2-(2\sqrt{ab})^2.$$

Ex. $\dfrac{\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}=\dfrac{\sqrt{2}(\sqrt{2}+\sqrt{3}+\sqrt{5})}{[(\sqrt{2}+\sqrt{3})-\sqrt{5}][(\sqrt{2}+\sqrt{3})+\sqrt{5}]}$

$$=\frac{2+\sqrt{6}+\sqrt{10}}{(\sqrt{2}+\sqrt{3})^2-5}=\frac{2+\sqrt{6}+\sqrt{10}}{2\sqrt{6}}$$

$$=\frac{(2+\sqrt{6}+\sqrt{10})\times\sqrt{6}}{2\sqrt{6}\times\sqrt{6}}$$

$$=\frac{2\sqrt{6}+6+2\sqrt{15}}{12}=\frac{\sqrt{6}+3+\sqrt{15}}{6}.$$

259. When applicable, the identities in Chapter IX. should be used in writing the quotient of two binomial surds.

Ex. $\dfrac{\sqrt{x^3}+\sqrt{y^3}}{\sqrt{x}+\sqrt{y}}\equiv\dfrac{(\sqrt{x})^3+(\sqrt{y})^3}{\sqrt{x}+\sqrt{y}}$

$$\equiv x-\sqrt{xy}+y.$$

Exercise 98.

Compute to three places of decimals:

1. $14 \div \sqrt{2}$.
2. $25 \div \sqrt{5}$.
3. $48 \div \sqrt{6}$.
4. $\sqrt{2} \div \sqrt{3}$.
5. $144 \div \sqrt{6}$.
6. $4 \div \sqrt{243}$.

Rationalize the denominator and simplify:

7. $3\sqrt{3}/(2\sqrt{2})$.
8. $\sqrt{15}/\sqrt{(3/5)}$.
9. $\sqrt{21}/\sqrt{(7/3)}$.
10. $10/\sqrt[3]{5}$.
11. $12/\sqrt[4]{3}$.
12. $2\sqrt[3]{6}/\sqrt{2}$.
13. $3\sqrt{2}/\sqrt[3]{9}$.
14. $\sqrt[3]{20}/(3\sqrt[6]{16})$.
15. $\sqrt{a}/\sqrt[3]{a}$.
16. $\sqrt{a}/\sqrt[4]{a}$.
17. $\sqrt[3]{x^2}/\sqrt[4]{x^3}$.
18. $\sqrt{(ax)}/\sqrt[4]{x}$.

19. $\dfrac{2\sqrt{5}}{\sqrt{5}+\sqrt{3}}$.

20. $\dfrac{15+14\sqrt{3}}{15-2\sqrt{3}}$.

21. $\dfrac{\sqrt{5}+3\sqrt{3}}{2\sqrt{5}-\sqrt{3}}$.

22. $\dfrac{\sqrt{6}-3\sqrt{12}}{2\sqrt{6}+\sqrt{12}}$.

23. $\dfrac{2\sqrt{3}+3\sqrt{2}}{5+2\sqrt{6}}$.

24. $\dfrac{\sqrt{9+x^2}-3}{\sqrt{9+x^2}+3}$.

25. $\dfrac{y^2}{x+\sqrt{x^2-y^2}}$.

26. $\dfrac{1}{1+\sqrt{2}+\sqrt{3}}$.

27. $\dfrac{3}{2+\sqrt{3}+\sqrt{5}}$.

28. $\dfrac{1+\sqrt{3}+\sqrt{6}}{1+\sqrt{2}-\sqrt{3}}$.

29. $\dfrac{\sqrt{3}}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$.

30. $\dfrac{3}{\sqrt{2}-\sqrt{6}-\sqrt{7}}$.

31. $\dfrac{\sqrt{x-2}+\sqrt{x}}{\sqrt{x-2}-\sqrt{x}}$.

32. $\dfrac{a+\sqrt{a^2+3}}{a-\sqrt{a^2+3}}$.

33. $\dfrac{\sqrt{a-b}-\sqrt{a+b}}{\sqrt{a-b}+\sqrt{a+b}}$.

34. $\dfrac{3+4\sqrt{3}}{\sqrt{6}+\sqrt{2}-\sqrt{5}}$.

Write each of the following quotients:

35. $(a-x) \div (\sqrt{a} - \sqrt{x})$.

36. $(ax - ay)/(\sqrt{x} + \sqrt{y})$.

37. $(1 - 1/x) \div (1 + 1/\sqrt{x})$.

38. $(a/b - x/y) \div [\sqrt{(a/b)} + \sqrt{(x/y)}]$.

39. $(\sqrt{a^3} - \sqrt{b^3}) \div (\sqrt{a} - \sqrt{b})$.

40. $(x\sqrt{x} - y\sqrt{y}) \div (\sqrt{x} + \sqrt{y})$.

Rationalize the denominator of each fraction:

41. $\dfrac{y^2}{x + \sqrt{x^2 - y^2}}$.

42. $\dfrac{x^2}{\sqrt{x^2 + a^2} + a}$.

43. $\dfrac{\sqrt{10} + \sqrt{5} + \sqrt{3}}{\sqrt{3} + \sqrt{10} - \sqrt{5}}$.

44. $\dfrac{\sqrt{1 + x^2} - \sqrt{1 - x}}{\sqrt{1 + x^2} + \sqrt{1 - x}}$.

45. $\dfrac{2\sqrt{a + b} - 3\sqrt{a - b}}{2\sqrt{a + b} - \sqrt{a - b}}$.

46. $\dfrac{(\sqrt{3} + \sqrt{5})(\sqrt{5} + \sqrt{2})}{\sqrt{2} + \sqrt{3} + \sqrt{5}}$.

260. **A root of a monomial surd** is found by applying the law

$$\sqrt[s]{\sqrt[q]{a}} \equiv \sqrt[sq]{a}. \qquad \text{§ 227}$$

Ex. 1. $\sqrt[3]{\sqrt{7}} = \sqrt[6]{7}$.

Observe that $\sqrt[s]{\sqrt[q]{a}} \equiv \sqrt[q]{\sqrt[s]{a}}$, since each member $\equiv \sqrt[sq]{a}$.

Ex. 2. $\sqrt{\sqrt[5]{(4x^2)}} \equiv \sqrt[5]{\sqrt{(4x^2)}} \equiv \sqrt[5]{(2x)}$.

Exercise 99.

Simplify each of the following expressions:

1. $\sqrt[3]{\sqrt[5]{(27a^3)}}$.
2. $\sqrt{\sqrt[7]{(9x^2)}}$.
3. $\sqrt[3]{\sqrt{(a^3)}}$.
4. $\sqrt[3]{\sqrt[4]{(x^6y^9)}}$.
5. $\sqrt[3]{(x\sqrt{x})}$.
6. $\sqrt{\sqrt[3]{(25x^2y^4/16)}}$.
7. $\sqrt[5]{(x^2\sqrt{x})}$.
8. $\sqrt{(2/\sqrt[3]{2})}$
9. $\sqrt[3]{(x^2/\sqrt{x})}$.
10. $\sqrt[n-1]{(x/\sqrt[n]{x})}$.
11. $\sqrt[n-1]{(x^2/\sqrt[n]{x^2})}$.
12. $\sqrt[3]{(bc/\sqrt[4]{bc})}$.

PROPERTIES OF QUADRATIC SURDS.

261. *If* $$x+\sqrt{y}=a+\sqrt{b}, \tag{1}$$
where x *and* a *are rational numbers, and* $\sqrt{y}$ *and* $\sqrt{b}$ *are surd numbers; then,*
$$x=a \text{ and } y=b. \tag{2}$$

Proof. Transposing a and squaring, from (1) we obtain
$$(x-a)^2+2(x-a)\sqrt{y}+y=b.$$
$$\therefore\ 2(x-a)\sqrt{y}=(b-y)-(x-a)^2. \tag{3}$$

Since a surd number cannot equal a rational number, equation (3) is satisfied when and only when $x=a$ and $y=b$.

262. *A quadratic surd number cannot be equal to the sum of a rational number, other than zero, and another quadratic surd number.*

Proof. Let $$\sqrt{b}=x+\sqrt{y}, \tag{1}$$
where x is a rational number and $\sqrt{b}$ and $\sqrt{y}$ are surd numbers; then, by § 261, we have
$$x=0 \text{ and } y=b. \tag{2}$$

263. Square root of the binomial surd $a+\sqrt{b}$.

Suppose $$\sqrt{a\pm\sqrt{b}}=\sqrt{x}\pm\sqrt{y}. \tag{1}$$

Square, $$a\pm\sqrt{b}=x+y\pm2\sqrt{xy}.$$

Hence, by § 261, we have
$$x+y=a,\ 2\sqrt{xy}=\sqrt{b};$$
or $$x+y=a,\quad 4xy=b. \tag{c}$$

Solving system (c) for x and y, and substituting their values in (1), we obtain the value of $\sqrt{(a\pm\sqrt{b})}$.

We shall here consider only those cases in which system (c) can be solved by inspection.

Ex. 1. Find the square root of $18 + 8\sqrt{5}$.

Assume $\sqrt{18 + 8\sqrt{5}} = \sqrt{x} + \sqrt{y}$. (1)

Square $18 + 8\sqrt{5} = x + y + 2\sqrt{xy}$.

$\therefore\ x + y = 18$, and $2\sqrt{xy} = 8\sqrt{5}$; § 261

or $x + y = 18$, and $xy = 80$. (a)

By inspection we see that one solution of system (a) is

$$x = 8,\ y = 10.$$

$$\therefore\ \sqrt{18 + 8\sqrt{5}} = \sqrt{8} + \sqrt{10} = 2\sqrt{2} + \sqrt{10}.$$

Ex. 2. Extract the square root of $83 - 12\sqrt{35}$.

Assume $\sqrt{83 - 12\sqrt{35}} = \sqrt{x} - \sqrt{y}$.

Square $83 - 12\sqrt{35} = x + y - 2\sqrt{xy}$.

$\therefore\ x + y = 83$, and $2\sqrt{xy} = 12\sqrt{35}$,

or $x + y = 83$, and $xy = 1260$.

By inspection, $x = 63$, and $y = 20$.

$$\therefore\ \sqrt{83 - 12\sqrt{35}} = \sqrt{63} - \sqrt{20} = 3\sqrt{7} - 2\sqrt{5}.$$

By taking $x = 20$ and $y = 63$, we would obtain the negative root of the given number.

Exercise 100.

Find the square root of the binomial surds:

1. $6 + \sqrt{20}$.	6. $11 - 2\sqrt{30}$.	11. $4\frac{1}{3} - \frac{4}{3}\sqrt{3}$.
2. $12 - 6\sqrt{3}$.	7. $7 - 2\sqrt{10}$.	12. $17 - 2\sqrt{66}$.
3. $16 + 6\sqrt{7}$.	8. $17 - 12\sqrt{2}$.	13. $19 + 8\sqrt{3}$.
4. $13 - 2\sqrt{42}$.	9. $47 - 4\sqrt{33}$.	14. $11 + 4\sqrt{6}$.
5. $28 - 5\sqrt{12}$.	10. $19 + 4\sqrt{22}$.	15. $15 - 4\sqrt{14}$.

CHAPTER XVIII

IMAGINARY AND COMPLEX NUMBERS

264. Quality-units $\sqrt{-1}$ and $-\sqrt{-1}$. As we have seen in § 219, an even root of a negative number, as $\sqrt{-2}$, cannot be a positive or a negative number, and therefore is not as yet included in our number system.

To give a meaning to such expressions as $\sqrt{-1}$ and $\sqrt{-2}$, we assume the identity

$$(\sqrt[n]{u})^n \equiv u \tag{1}$$

to hold when u is negative and n is even (§§ 213, 239).

Thus, $\sqrt{-2}$ denotes that number whose square is -2.

An important particular case of (1) is

$$(\sqrt{-1})^2 = -1. \tag{2}$$

Since any power or root of $+1$ or -1, heretofore obtained, is a quality-unit, we call $\sqrt{-1}$ a *quality-unit.*

That is, $\sqrt{-1}$ *is a* **quality-unit** *whose square is* -1.

Squaring both members of (2), we obtain

$$(\sqrt{-1})^4 = +1; \tag{3}$$

that is, *the fourth power of* $\sqrt{-1}$ *is equal to* $+1$.

Again, the *opposite* of the quality-unit $\sqrt{-1}$ is $-\sqrt{-1}$.

Also, $$(\sqrt{-1})^3 = (\sqrt{-1})^2 \cdot \sqrt{-1} = -\sqrt{-1}; \tag{4}$$

that is, *the cube of* $\sqrt{-1}$ *is equal to its opposite,* $-\sqrt{-1}$.

The quality-units $\sqrt{-1}$ and $-\sqrt{-1}$ involve the idea of the *arithmetic one* and that of *oppositeness* to each other.

Observe carefully the above relations of the quality-unit $\sqrt{-1}$ to -1, $+1$, and $-\sqrt{-1}$.

The quality-units $\sqrt{-1}$ and $-\sqrt{-1}$ are called **imaginary units.**

The units $\sqrt{-1}$ and $-\sqrt{-1}$ are for brevity often denoted by i and $-i$, i being used as a numeral.

NOTE. The word *imaginary*, as here used, must not be understood as implying that the units i and $-i$ are any less real than $+1$ and -1. The expression $\sqrt{-1}$ was called imaginary when it first made its appearance in Algebra before its meaning and uses were understood. The name is unfortunate, but with the above explanation we shall use it.

265. Imaginary numbers. Any arithmetic multiple of the imaginary unit i or $-i$ is called an *imaginary number.*

An imaginary number is *commensurable* or *incommensurable*, according as its arithmetic factor is commensurable or incommensurable.

E.g., $i3$, $i(3/5)$, $-i7$ are commensurable, while $i\sqrt{2}$ and $-i\sqrt{(2/3)}$ are incommensurable imaginary numbers.

266. Multiplication by the imaginary unit $\sqrt{-1}$, or i, is defined by assuming the commutative law; that is, a being any number, we assume that

$$a \times \sqrt{-1} \equiv \sqrt{-1} \times a, \text{ or } a \times i = ia.$$

267. Since $i \cdot a \equiv a \cdot i$, the imaginary number $i \cdot a$ or $-i \cdot a$ can be written ai or $-ai$.

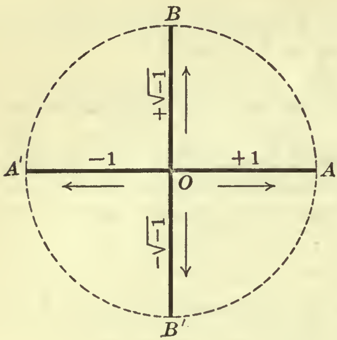
Imaginary whole numbers form the following series:

$$\cdots, \quad -3i, \quad -2i, \quad -i, \quad 0, \quad i, \quad 2i, \quad 3i, \quad \cdots.$$

Observe that the one and only number which is common to the series of real and imaginary numbers is 0.

268. Geometric representation of quality-units.

A *directed* line is a line whose *direction* and *length* are both considered. To represent geometrically quality-numbers we use directed lines. Of the directed line OA', O is the *origin* and A the *end*.



Let $OA = +1$; then $OA' = -1$.

Substituting these values in

$$(+1) \cdot \sqrt{-1} \cdot \sqrt{-1} = -1,$$

we obtain

$$OA \cdot \sqrt{-1} \cdot \sqrt{-1} = OA'.$$

That is, multiplying OA by $\sqrt{-1}$ twice in succession reverses its direction; hence we can assume that multiplying OA by $\sqrt{-1}$ twice in succession revolves OA through two right angles in the plane ABA' and in a direction opposite to that of the hands of a clock.

Hence multiplying OA by $\sqrt{-1}$ once would revolve OA through one right angle in this direction; that is,

$$OA \cdot \sqrt{-1} = OB. \qquad (1)$$

But $$OA \cdot \sqrt{-1} = (+1) \cdot \sqrt{-1} = \sqrt{-1}. \qquad (2)$$

From (1), (2), $$OB = \sqrt{-1}, \text{ or } i.$$

$$\therefore\ OB' = -OB = -i.$$

Hence, if the *primary* quality-unit $+1$ is represented by the directed line OA, the quality-units -1, i, and $-i$ will be represented by the directed lines OA', OB, and OB', respectively.

As the lines OB and OB' are just as real as the lines OA and OA', so the quality-units i and $-i$ are just as real as $+1$ and -1.

Arithmetic multiples of i and $-i$ can be represented by distances along the lines OB and OB' or their extensions, just as multiples of $+1$ and -1 are represented by distances along the lines OA and OA' or their extensions.

Again, if in a football game we denote the forces exerted in the direction OA by positive real numbers; then negative real numbers will denote the forces exerted in the opposite direction OA', positive

imaginary numbers will denote the forces exerted in the direction OB, and negative imaginary numbers will denote the forces exerted in the direction OB'.

To express by numbers the magnitudes and *directions* of the many other forces in the game we need still further to enlarge our concept of quality-numbers, as is done in § 285.

269. Since imaginary numbers are simply arithmetic multiples of the units i and $-i$, they are added and subtracted the same as real numbers.

That is, $$ai \pm bi \equiv (a \pm b)i, \qquad (1)$$

which is the converse of the distributive law.

Ex. 1. $4i + 6i = (4 + 6)i = 10i.$

Ex. 2. $(7/3)i - (5/3)i = (7/3 - 5/3)i = (2/3)i.$

270. When the imaginary unit is a factor of a product, the distributive law follows from its converse in § 269, and the associative law follows from the commutative law in § 266;

that is, $$(a \pm b)i \equiv ai \pm bi, \qquad \text{§ 269}$$

and $$ai \cdot bi \equiv i^2ab \equiv -ab. \qquad \text{§ 267}$$

271. Powers of i. From § 264, we have

$$i^2 = -1,\ i^3 = -i,\ i^4 = +1. \qquad (1)$$

Ex. 1. $i^7 = i^4 \cdot i^3 = (+1)(-i) = -i.$ by (1)

Ex. 2. $i^{10} = (i^4)^2 i^2 = (+1)^2(-1) = -1.$ by (2)

Ex. 3. $i^{13} = (i^4)^3 i = (+1)^3 i = i.$ by (1)

If n is any positive integer including zero, we have

$$i^{4n} \equiv (i^4)^n \equiv (+1)^n \equiv +1; \qquad (2)$$

$$i^{4n+1} \equiv i^{4n}i \equiv i; \qquad \text{by (2)}$$

$$i^{4n+2} \equiv i^{4n}i^2 \equiv -1; \qquad \text{by (1), (2)}$$

$$i^{4n+3} \equiv i^{4n}i^3 \equiv -i. \qquad \text{by (1), (2)}$$

Hence, *any even power of* i *is* $+1$ *or* -1, *and any odd power of* i *is* i *or* $-i$.

272. *The square root of any negative number is an imaginary number;*

that is, $$\sqrt{{}^{-}a} \equiv \sqrt{a} \cdot i. \qquad (1)$$

Proof. By the commutative and associative laws we have

$$(\sqrt{a} \cdot \sqrt{-1})^2 \equiv (\sqrt{a})^2(\sqrt{-1})^2 \equiv {}^{-}a; \qquad \S\ 271$$

hence, $$\sqrt{a}\sqrt{-1} \equiv \sqrt{{}^{-}a}, \text{ or conversely (1).} \qquad \S\ 221$$

E.g., $$\sqrt{-16} = i \cdot \sqrt{16} = 4\,i, \text{ or } 4\sqrt{-1}.$$

and $$\sqrt{-a^2} \equiv i \cdot \sqrt{a^2} \equiv ai, \text{ or } a\sqrt{-1}.$$

273. To add or subtract imaginary numbers given in the form $\sqrt{{}^{-}a}$, we first reduce them to the *type-form*, $\sqrt{a} \cdot i$.

Ex. 1. $$\sqrt{-49} + \sqrt{-81} - \sqrt{-36} = 7\,i + 9\,i - 6\,i$$
$$= (7 + 9 - 6)i = 10\,i, \text{ or } 10\sqrt{-1}.$$

Ex. 2. $$\sqrt{-9\,a^2} + \sqrt{-4\,b^2} - \sqrt{-7\,c^2} \equiv 3\,a \cdot i + 2\,b \cdot i - c\sqrt{7} \cdot i$$
$$\equiv (3\,a + 2\,b - c\sqrt{7})i.$$

274. From the commutative and associative laws we have the following principle:

The product of two or more quality-numbers is equal to the product of their quality-units into the product of their arithmetic values.

Ex. 1. $\sqrt{-5}(-\sqrt{11}) = i \cdot {}^{-}1 \cdot \sqrt{5} \cdot \sqrt{11} = -\,i\sqrt{55}$, or $-\sqrt{-55}$.

Ex. 2. $\sqrt{-3} \cdot \sqrt{-7} = i^2\sqrt{3} \cdot \sqrt{7} = -\sqrt{21}$.

Ex. 3. $\sqrt{-2} \cdot \sqrt{-3} \cdot \sqrt{-5} = i^3\sqrt{2} \cdot \sqrt{3} \cdot \sqrt{5} = -\,i\sqrt{30}$, or $-\sqrt{-30}$.

Ex. 4. $\sqrt{{}^{-}a} \cdot \sqrt{{}^{-}b} \cdot \sqrt{{}^{-}c} \cdot \sqrt{{}^{-}n} \equiv i^4\sqrt{a}\sqrt{b}\sqrt{c}\sqrt{n} \equiv \sqrt{abcd}$.

Exercise 101.

Simplify each of the following expressions:

1. $\sqrt{-36}+\sqrt{-49}-\sqrt{-100}$.

2. $\sqrt{-4}-\sqrt{-9}+\sqrt{-16}$.

3. $\sqrt{-81}-\sqrt{-64}-\sqrt{-121}$.

4. $\sqrt{-9a^2}-\sqrt{-4a^2}-\sqrt{-16a^2}$.

5. $\sqrt{-36b^2}-\sqrt{-49b^2}+\sqrt{-81b^2}$.

6. $\sqrt{-(x+a)^2}+\sqrt{-(x-a)^2}$.

7. $3\sqrt{-a}+7\sqrt{-c}-11\sqrt{-b}+2\sqrt{-(m^2n^2)}$.

8. $\sqrt{2}\cdot\sqrt{-3}$.

9. $\sqrt{3}\cdot\sqrt{-5}$.

10. $\sqrt{-5}\cdot\sqrt{-11}$.

11. $\sqrt{-2}\cdot\sqrt{-7}$.

12. $\sqrt{-4}\cdot\sqrt{-9}$.

13. $\sqrt{-9}\cdot\sqrt{16}$.

14. $\sqrt{-2}\cdot\sqrt{-3}\cdot\sqrt{-7}$.

15. $\sqrt{6}\cdot\sqrt{-2}\cdot\sqrt{-5}$.

16. $\sqrt{5}\cdot\sqrt{-2}\cdot\sqrt{-3}\cdot\sqrt{-4}$.

17. $\sqrt{-a^2}\cdot\sqrt{-b^2}\cdot\sqrt{-c^2}$.

18. $\sqrt{-3ax}\cdot\sqrt{-3bx}\cdot\sqrt{-4ab}$.

19. $(\sqrt{-2}+\sqrt{-5})(\sqrt{-3}+\sqrt{-7})$.

20. $(2\sqrt{-3}+3\sqrt{-5})(5\sqrt{-3}-2\sqrt{-5})$.

21. $(\sqrt{-x}+\sqrt{-y})(\sqrt{-x}-\sqrt{-y})$.

22. $\sqrt{-a^2}$. 23. $(\sqrt{-b})^2$. 24. $\sqrt{-y^4}$. 25. $(\sqrt{-y})^4$.

275. Quotient of one quality-unit by another.

$$i\div i\equiv 1;\ i\div(+1)\equiv i;\ i\div(-1)\equiv -i; \qquad \S\ 84$$

$$+1\div i\equiv i^4\div i\equiv i^3=-i; \qquad \S\ 264$$

$$-1\div i\equiv i^2\div i,\ \equiv i.$$

E.g., $1 \div i^5 \equiv 1 \div i \equiv -i$; $-1 \div i^6 \equiv -1 \div (-1) \equiv 1$;

$$i \div i^7 \equiv i \div (-i) \equiv -1.$$

276. From the commutative and associative laws we have the following principle:

The quotient of two quality-numbers is equal to the quotient of their quality-units into the quotient of their arithmetic values.

Ex. 1. $\dfrac{\sqrt{-5}}{\sqrt{-10}} = \dfrac{i}{i} \cdot \dfrac{\sqrt{5}}{\sqrt{10}} = \dfrac{\sqrt{2}}{2}.$

Ex. 2. $\dfrac{\sqrt{-3}}{\sqrt{7}} = \dfrac{i}{{}^{+}1} \cdot \dfrac{\sqrt{3}}{\sqrt{7}} = i\dfrac{\sqrt{21}}{7}$, or $\dfrac{\sqrt{-21}}{7}.$

Ex. 3. $\sqrt{{}^{-}a}/\sqrt{{}^{-}b} \equiv (i/i)(\sqrt{a}/\sqrt{b}) \equiv \sqrt{a/b}.$

Ex. 4. $\sqrt{{}^{+}a}/\sqrt{{}^{-}b} \equiv ({}^{+}1/i)(\sqrt{a}/\sqrt{b}) \equiv -i\sqrt{a/b}$, or $-\sqrt{{}^{-}(a/b)}.$

Exercise 102.

Perform the operation of division in

1. $i^3 \div i.$
2. $i^5 \div i.$
3. $1 \div i^3$
4. $-1 \div i^2.$
5. $i \div i^3.$
6. $\sqrt{-14} \div \sqrt{-2}.$
7. $\sqrt{-16} \div \sqrt{-4}.$
8. $\sqrt{-15} \div \sqrt{-3}.$
9. $\sqrt{{}^{-}a^2} \div \sqrt{{}^{-}a}.$
10. $\sqrt{{}^{-}(xy)} \div \sqrt{x}.$
11. $\sqrt{{}^{-}a^2} \div \sqrt{{}^{-}b^2}.$
12. $(\sqrt{-12} - \sqrt{-15}) \div \sqrt{-3}.$
13. $(\sqrt{{}^{-}a} + \sqrt{{}^{-}b}) \div \sqrt{{}^{-}c}.$
14. $(\sqrt{16} - \sqrt{8}) \div \sqrt{-2}.$
15. $i \div i^{13}.$
16. $-i \div i^{17}.$
17. $i^3 \div i^{27}.$
18. $i^{12} \div i^{33}.$

COMPLEX NUMBERS.

277. The sum of a real number and an imaginary number is called a **complex number**, as $4 \pm 5i$, $7 \pm 3i$.

The general expression for a complex number is evidently $a + bi$, where a and b are any real numbers.

When $b = 0$, $a + bi = a$, a real number.
When $a = 0$, $a + bi = bi$, an imaginary number.

278. We define addition of complex numbers by assuming the commutative, and therefore the associative, law of addition for real and imaginary numbers.

Hence, *in adding or subtracting complex numbers the real parts can be added or subtracted by themselves and the imaginary parts by themselves.*

That is, $(a + bi) \pm (c + di) \equiv (a \pm c) + (b \pm d)\,i.$ (1)

Ex. When is the second member of (1) a complex number? When an imaginary number? When a real number?

279. *If two complex numbers are equal, their real parts are equal, and their imaginary parts are equal.*

Proof. Let $a + bi = c + di,$ (1)

where a, b, c, d are all real numbers.

Transposing, $a - c = di - bi.$ (2)

But, if a real number is equal to an imaginary number, each is zero (§ 267); hence,

$$a - c = 0, \text{ or } a = c,$$

and $$di - bi = 0, \text{ or } d = b.$$

An important case of this theorem is the following:
If $a + bi = 0$, then $a = 0$ and $b = 0$.

280. Two complex numbers which differ only in the signs before their imaginary terms are called **conjugate complex numbers,** as $a + bi$ and $a - bi$.

Since $(a + bi) + (a - bi) \equiv 2\,a,$

the sum of two conjugate complex numbers is real.

281. Multiplication by a complex number is defined by assuming the distributive law; that is,

$$(a+bi)(c+di) \equiv ac + adi + bci + bdi^2$$
$$\equiv (ac - bd) + (ad + bc)i. \qquad (1)$$

Before multiplying one complex number by another it is convenient to reduce each to the type-form $a + bi$.

Ex. $(3+\sqrt{-5})(4-\sqrt{-3}) = (3+\sqrt{5}\cdot i)(4-\sqrt{3}\cdot i)$
$= 12 + (4\sqrt{5} - 3\sqrt{3})i + \sqrt{15}.$

282. From (1) in § 281, it follows that the product of two complex numbers is, in general, a complex number.

But, *the product of two conjugate complex numbers is real and positive.*

Proof. $(a+bi)(a-bi) \equiv a^2 - (bi)^2 \equiv a^2 + b^2.$

E.g., $(-3+\sqrt{-2})(-3-\sqrt{-2}) = (-3)^2 - (\sqrt{-2})^2 = 11.$

283. *The quotient of one complex number by another is, in general, a complex number.*

Proof.
$$\frac{a+bi}{c+di} \equiv \frac{(a+bi)(c-di)}{(c+di)(c-di)}$$
$$\equiv \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i. \qquad (1)$$

284. From (1) in § 283, it follows that when the divisor is a complex number, the quotient can be expressed as a complex number by multiplying both the dividend and divisor by the conjugate of the divisor.

Ex.
$$\frac{4+3i}{3-2i} = \frac{(4+3i)(3+2i)}{(3-2i)(3+2i)}$$
$$= \frac{6+17i}{9+4} = \frac{6}{13} + \frac{17}{13}i.$$

Exercise 103.

Find each of the following sums and products:

1. $(2+\sqrt{-4})+(3-\sqrt{-1})$.

2. $(3+\sqrt{-9})-(7-\sqrt{-16})$.

3. $(3-2i)+(6+5i)$.

4. $(8+\sqrt{-36})-(5+\sqrt{-25})$.

5. $(4+3i)-(3-4i)$.

6. $(2+\sqrt{-3})+(2-\sqrt{-3})$.

7. $(2+\sqrt{-4})(3-\sqrt{-9})$.

8. $(4+\sqrt{-16})(3-\sqrt{-25})$.

9. $(5+2\sqrt{-9})(2-3\sqrt{-4})$.

10. $(1+\sqrt{-7})(2-\sqrt{-16})$.

11. $(3+\sqrt{-5})(2-\sqrt{-3})$.

12. $(\sqrt{2}+\sqrt{-2})(\sqrt{3}-\sqrt{-3})$.

13. $(2+\sqrt{-3})(2-\sqrt{-3})$.

14. $(-4-\sqrt{-5})(-4+\sqrt{-5})$.

15. $(-7-\sqrt{-11})(-7+\sqrt{-11})$.

16. $(x\sqrt{-x}+y\sqrt{-y})(x\sqrt{-x}-y\sqrt{-y})$.

17. $(\sqrt{-7}+5\sqrt{-3})(\sqrt{-7}+3\sqrt{-2})$.

18. $(1+\sqrt{-3})^2$.

19. $(\sqrt{-3}+\sqrt{2})^2$

20. $(2+3ai)^2$.

21. $(2-3\sqrt{-2})^3$.

22. $(a+ci)^3$.

Reduce each of the following expressions to the *typical form* $a+bi$:

23. $\dfrac{1}{2-\sqrt{-3}}$.

24. $\dfrac{1+\sqrt{-1}}{1-\sqrt{-1}}$.

25. $\dfrac{4+\sqrt{-2}}{2-\sqrt{-2}}$.

26. $\dfrac{a+\sqrt{-x}}{a-\sqrt{-x}}$.

27. $\dfrac{1}{3-2\sqrt{-3}}$.

28. $\dfrac{3+4\sqrt{-5}}{2-3\sqrt{-5}}$.

29. $\dfrac{3+2\sqrt{-1}}{2-3\sqrt{-1}}$.

30. $\dfrac{1+3i}{3-4i}$.

31. $\dfrac{3+\sqrt{-5}}{4-\sqrt{-7}}$.

32. Show that

$[(-1+\sqrt{-3})/2]^3=+1$; also $[(-1-\sqrt{-3})/2]^3=+1$;

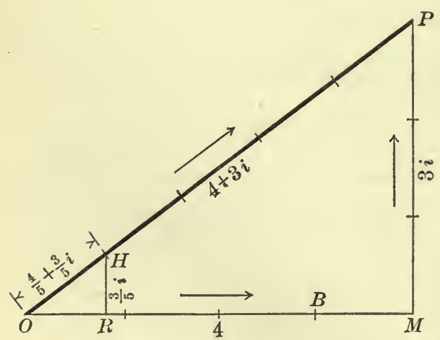
and that therefore there are at least three cube roots of $+1$.

33. In § 281, when is the product a complex number? When an imaginary number? When a real number?

34. In § 283, when is the quotient a complex number? When an imaginary number? When a real number?

285. Geometric representation of a complex number.

To find the sum of $+4$ and -1 geometrically, we lay off OM equal to 4 in the positive direction; then from the *end* of OM we lay off MB equal to 1 in the negative direction. The straight line OB, which extends from the *origin* O of the line OM to the *end* B of the line MB, is the sum of $+4$ and -1; that is, the sum is the *directed line drawn from the* origin *of the first line to the* end *of the second.*



Similarly, to construct the sum $4+3i$, we lay off OM equal to $+4$; at M, the end of OM, we erect a perpendicular and on it, in the

direction of positive imaginaries, lay off MP three units long; then OP, *i.e.* the *directed line* extending from the *origin* O of the first line to the *end* P of the second line, will represent the complex number $4+3i$.

From the right-angled triangle OMP we have

$$\text{the length of } OP = \sqrt{4^2+3^2} = 5.$$

If we take $OH =$ one unit long and draw HR parallel to MP, then from the similar triangles ORH and OMP we have $OR = 4/5$ and $RH = (3/5)i$, that is, the directed unit-line OH represents $4/5 + (3/5)i$, and OP, which represents $4+3i$, is 5 times this unit-line.

Hence the *arithmetic value* of $4+3i$ is 5, and its quality-unit is $4/5+(3/5)i$, which illustrates § 286.

Observe that the quality-unit $4/5+(3/5)i$ is obtained by dividing $4+3i$ by 5, and that $5=\sqrt{4^2+3^2}$.

In like manner we can represent any other complex number and its quality-unit.

286. The **arithmetic value**, or **modulus**, of the *complex number* $a+bi$ is the square root of the sum of the squares of a and b, or $\sqrt{a^2+b^2}$; and its quality-unit is $(a+bi)/\sqrt{a^2+b^2}$.

That is, $a+bi \equiv [(a+bi)/\sqrt{a^2+b^2}]\cdot\sqrt{a^2+b^2}$,

where the complex number $a+bi$ is written as the product of a quality-unit and an arithmetic number.

E.g. the *arithmetic* value, or *modulus*, of $4+3i$ is $\sqrt{4^2+3^2}$, or 5.

The modulus of $5-3i$ is $\sqrt{5^2+(-3)^2}$, or $\sqrt{34}$.

The modulus of $-1/2+\sqrt{-3}/2$ is $\sqrt{(1/2)^2+(\sqrt{3}/2)^2}$, or 1.

CHAPTER XIX

QUADRATIC EQUATIONS IN ONE UNKNOWN

287. By the principles of equivalence of equations in Chapter VII. we can derive from any quadratic equation in one unknown, as x, an *equivalent* equation of the *type-form*

$$ax^2 + bx + c = 0. \tag{A}$$

Observe that in (A), ax^2 is the sum of all the terms in x^2, bx is the sum of all the terms in x, and c is the sum of all the terms free from x.

E.g., from the quadratic equation

$$\frac{6x^2-3}{2} - \frac{5x^2-3}{4} = \frac{x^2-10x}{3}$$

we derive the equivalent equation

$$13x^2 - 40x - 9 = 0, \tag{1}$$

which is in the type-form.

Comparing (1) with (A) we have

$$a = 13,\ b = -40,\ c = -9.$$

If $a = 0$, (A) ceases to be a quadratic equation; hence in what follows we shall assume that a is not zero.

If neither b nor c is zero, (A) is called a *complete* quadratic equation.

If either b or c is zero, or if both are zero, (A) is called an *incomplete* quadratic equation. When $b = 0$, the incomplete equation is often called a *pure* or *binomial* quadratic equation.

E.g., equation (1) is a complete quadratic equation; while $3x^2 + 4x = 0$, $8x^2 + 9 = 0$, and $5x^2 = 0$ are incomplete, the last two being *pure*.

Ex. 1. In examples 20–30 of exercise 105 reduce each equation to an equivalent equation of the type-form (A), and state the values of a, b, and c in each.

Ex. 2. Solve the *incomplete* quadratic equation

$$ax^2 + bx = 0. \tag{1}$$

Factor $$x(ax + b) = 0. \tag{2}$$

Equation (2) is equivalent to the two linear equations,

$$x = 0, \ ax + b = 0.$$

Hence the roots of (2) or (1) are 0 and $-b/a$.

Remember that to solve any quadratic or higher equation we must first find its equivalent linear equations. Reread §§ 148 and 149.

288. Since $u^2 + mu + (m/2)^2 \equiv (u + m/2)^2$, § 137

The expression $\boldsymbol{u^2 + mu}$ *is made a perfect square by adding* $\boldsymbol{(m/2)^2}$, *or the square of one-half the coefficient of* $\boldsymbol{u}$.

The addition of $(m/2)^2$ is called **completing the square.**

E.g., $x^2 - 7x$ is made a perfect square by adding $(-7/2)^2$ or $(7/2)^2$;

that is, $$x^2 - 7x + 49/4 \equiv (x - 7/2)^2.$$

$4x^2 + 8x$, or $(2x)^2 + 4(2x)$, is made a perfect square by adding $(4/2)^2$, or 4;

that is, $$(2x)^2 + 4(2x) + 4 \equiv (2x + 2)^2.$$

289. Any quadratic equation can be solved by transposing all its terms to one member, factoring that member by *writing it as the difference of two squares*, and then putting each factor equal to zero.

The following examples will illustrate the method.

Ex. 1. Solve the pure quadratic equation

$$ax^2 + c = 0. \tag{1}$$

Divide by a, $$x^2 - (-c/a) = 0. \tag{2}$$

Factor, $$(x - \sqrt{-c/a})(x + \sqrt{-c/a}) = 0.$$

By § 149, $$x = \sqrt{-c/a}, \ x = -\sqrt{-c/a}.$$

Writing these two linear equations together, we have

$$x = \pm\sqrt{-c/a}. \qquad (3)$$

Ex. 2. Solve the complete quadratic equation

$$x^2 + 4x - 2 = 0. \qquad (1)$$

Add $4 - 4$, $\qquad x^2 + 4x + 4 - 6 = 0;$

or $\qquad (x+2)^2 - (\sqrt{6})^2 = 0. \qquad (2)$

Factor, $\qquad (x + 2 - \sqrt{6})(x + 2 + \sqrt{6}) = 0.$

By § 149, $\qquad x + 2 = \sqrt{6},\ x + 2 = -\sqrt{6}.$

Writing these two linear equations together, we have

$$x + 2 = \pm\sqrt{6}. \qquad (3)$$

$$\therefore\ x = -2 \pm \sqrt{6}.$$

Observe that in each example the two linear equations in (3) can be obtained from (2) by transposing the known term and then extracting the square root of both members, writing the double sign $\pm$ with one member. The principle of equivalence of equations which this illustrates is proved in the next article.

290. Square root. *If the square root of both members of an equation is extracted, and the double sign $\pm$ is written before one member, the two derived equations (when rational in the unknown) will together be equivalent to the given equation.*

Proof. Let the given equation be

$$A^2 = B^2, \qquad (1)$$

where A and B are rational in the unknown.

Transpose, $\qquad A^2 - B^2 = 0.$

Factor, $\qquad (A - B)(A + B) = 0. \qquad (2)$

By §§ 149 and 106, (2) is equivalent to the two equations

$$A = \pm B. \qquad (3)$$

Equations (3) can be obtained from (1) by extracting the square root of both members and writing the double sign $\pm$ with the second member; hence the theorem.

The following examples illustrate how this principle, which is proved by factoring, abbreviates the work of finding the *two linear* equations, which are equivalent to a given quadratic equation.

Ex. 1. Solve $$x^2 + 32 - 10x = 0. \tag{1}$$

Transpose 32, $$x^2 - 10x = -32.$$

Add $(10/2)^2$, $$x^2 - 10x + 25 = 25 - 32 = -7.$$

Extract square root, $$x - 5 = \pm\sqrt{-7}.$$

$$\therefore x = 5 \pm \sqrt{-7}. \tag{2}$$

By §§ 106 and 290, no root is either introduced or lost in passing from (1) to (2); hence the roots of (1) are $5 + \sqrt{-7}$ and $5 - \sqrt{-7}$.

Ex. 2. Solve $$\frac{x^2+9}{4} = \frac{x^2+1}{5}. \tag{1}$$

Multiply by 20, $$5x^2 + 45 = 4x^2 + 4.$$

Transpose, $$x^2 = -41.$$

Extract square root, $$x = \pm\sqrt{-41}. \tag{2}$$

By §§ 106, 108, and 290, no root is either introduced or lost in passing from (1) to (2); hence the roots of (1) are $+\sqrt{-41}$ and $-\sqrt{-41}$.

Ex. 3. Solve $$2x^2 = 7x + 11.$$

Transpose $7x$, $$2x^2 - 7x = 11.$$

Multiply by 2, $$(2x)^2 - 7(2x) = 22.$$

Complete square, $$(2x)^2 - 7(2x) + 49/4 = 22 + 49/4 = 137/4.$$

Extract square root, $$2x - 7/2 = \pm\sqrt{137}/2.$$

$$\therefore x = (7 \pm \sqrt{137})/4.$$

Hence, to solve a quadratic equation we can proceed as follows:

Reduce the equation to the form $ax^2 + bx = -c$.

If the *term* in x^2 is not a perfect square, multiply (or divide) both members by a number which will make it a perfect square.

Add to both members what is necessary to complete the square of the unknown member.

Extract the square root of each member, writing the double sign $\pm$ before the known member.

Solve the two derived linear equations.

Exercise 104.

Solve each of the following equations:

1. $x^2 + 1 = 4x$.
2. $x^2 - 2x = 4$.
3. $x^2 + 5 = 8x$.
4. $x^2 + 2x = 2$.
5. $x^2 + 6x = -3$.
6. $4x^2 + 4x = 11$.
7. $9x^2 + 6x = 17$.
8. $4x^2 - 4x - 7 = 0$.
9. $3x^2 - 6x + 2 = 0$.
10. $5x^2 - 6x + 11 = 0$.
11. $3x^2 + 4x + 7 = 0$.
12. $2x^2 - 6x + 10 = 0$.
13. $5x^2 + 8x + 21 = 0$.
14. $2x^2 - 5x + 15 = 0$.
15. $2x^2 - 3ax + 2a^2 = 0$.
16. $(x - 7)^2 = 49(x + 2)^2$.

When *both* members are perfect squares in the unknown, as in example 16 (or can be made so, as in some of the examples which follow), the first step is to extract the square root of both members.

17. $(x + 2)^2 = 4(x - 1)^2$.
18. $(x + 6)^2 = 16(x - 6)^2$.
19. $(x + 8)^2 = 9x^2$.
20. $x^2 - 3ax + 2a^2 = 0$.
21. $x^2 + 2ax = b^2 + 2ab$.
22. $x^2 + 2ab = b^2 + 2ax$.
23. $4x^2 + 4ax = b^2 - a^2$.
24. $x^2 + 3a^2 = 4ax$.

291. *To solve the general quadratic equation,*

$$\boldsymbol{ax^2 + bx + c = 0}, \qquad (A)$$

we proceed just as with the particular equations above.

Transpose c, $\qquad ax^2 + bx = -c$.

Multiply by $4a$ instead of a, to avoid fractions in (1) and (2),

$$(2ax)^2 + 2b(2ax) = -4ac.$$

Add b^2, $\quad (2ax)^2 + 2b(2ax) + b^2 = b^2 - 4ac. \quad (1)$

Extract square root, $\quad 2ax + b = \pm\sqrt{b^2 - 4ac}. \quad (2)$

Hence, $\quad \boldsymbol{x = (-b \pm \sqrt{b^2 - 4ac})/(2a)}. \quad (B)$

By §§ 106, 108, and 290, no root is either introduced or lost in passing from (A) to (B); hence the roots of (A) are given in (B).

Let b' and c' denote the values of b and c when $a = 1$.

Then when $a = 1$, equations (A) and (B) become

$$\boldsymbol{x^2 + b'x + c' = 0,} \qquad (A')$$

and

$$\boldsymbol{x = -b'/2 \pm \sqrt{(b'/2)^2 - c}.} \qquad (B')$$

By § 287, any quadratic equation can be reduced to an equivalent equation of the form (A); hence, *a quadratic equation in one unknown has two, and only two, roots.*

292. Solution by formula. Instead of repeating the process in § 291 with every quadratic equation, we should hereafter find the values of a, b, and c when the equation is reduced to the *type-form* (A), and substitute these values in the two equations (B),

$$x = (-b \pm \sqrt{b^2 - 4ac})/(2a). \qquad (B)$$

Ex. 1. Solve $2x^2 - 3x + 5 = 0$.

Here $a = 2,\ b = -3,\ c = 5$.

Substituting these values in equations (B), we obtain

$$x = (3 \pm \sqrt{9 - 40})/4 = (3 \pm \sqrt{-31})/4.$$

Ex. 2. Solve $-3x^2 = 3k - 2ax$.

Here $a = -3,\ b = 2a,\ c = -3k$.

Substituting these values in equations (B), we obtain

$$x = (-2a \pm \sqrt{4a^2 - 36k})/(-6)$$
$$= (a \mp \sqrt{a^2 - 9k})/3.$$

293. Equations (B') of § 291 afford the following simple rule for writing out the two roots of an equation in the form

$$x^2 + b'x + c' = 0.$$

The two roots are equal to minus one-half the coefficient of x *plus and minus the square root of the binomial, the square of one-half the coefficient of* x *minus the known term.*

Ex. 3. Solve $x^2 + 4x + 7 = 0$, by the rule given above.

$$x = -2 \pm \sqrt{2^2 - 7} = -2 \pm \sqrt{-3}.$$

Ex. 4. Solve $x^2 - 6x - 8 = 0$,

$$x = 3 \pm \sqrt{(-3)^2 - (-8)} = 3 \pm \sqrt{17}.$$

Exercise 105.

Solve each of the following equations by § 293:

1. $x^2 - 2x = 1$.
2. $x^2 + 8x + 5 = 0$.
3. $x^2 + 4x = 1$.
4. $x^2 + 18 = 10x$.
5. $x^2 + 3 = 2x$.
6. $x^2 + 11 = 4x$.
7. $x^2 + 31 = 10x$.
8. $x^2 + 6x + 11 = 0$.
9. $x^2 + 10x + 32 = 0$.
10. $x^2 + 52 = 14x$.
11. $x^2 + 2x = 1$.
12. $x^2 = 4x - 18$.

Solve each of the following equations by § 292:

13. $3x^2 + 121 = 44x$.
14. $25x = 6x^2 + 21$.
15. $8x^2 + x = 30$.
16. $3x^2 + 35 = 22x$.
17. $x + 22 = 6x^2$.
18. $15 = 17x + 4x^2$.
19. $21 + x = 2x^2$.
20. $9x^2 - 143 = 6x$.
21. $12x^2 = 29x - 14$.
22. $20x^2 = 12 - x$.
23. $15x^2 - 2ax = a^2$.
24. $21x^2 = 2ax + 3a^2$.

Solve each of the following equations by the method best suited to it:

25. $9x^2 - 6ax = a^2 - b^2$. 26. $a(x^2+1) = x(a^2+1)$.

27. $a(x^2-1) + x(a^2-1) = 0$.

28. $x^2 - 2(a-b)x + b^2 = 2ab$.

29. $(b-c)x^2 + (c-a)x = b-a$.

30. $(a+b)x^2 + cx = a+b+c$.

31. $abx^2 - (a^2+b^2)x + ab = 0$.

32. $(a^2-b^2)(x^2-1) = 4abx$.

33. $(b^2-a^2)(x^2+1) = 2(a^2+b^2)x$.

34. $(a-x)^3 + (x-b)^3 = (a-b)^3$.

35. $(x-a+2b)^3 - (x-2a+b)^3 = (a+b)^3$.

294. Discussion of the roots, $(-b \pm \sqrt{b^2-4ac})/(2a)$, *when a, b, c are real.*

(i) If $b^2 - 4ac > 0$, the two roots will be *real* and *unequal.*

(ii) If $b^2 - 4ac = 0$, the two roots will be *real* and *equal.*

(iii) If $b^2 - 4ac < 0$, the two roots will be *imaginary* or *complex.*

(iv) If $b = 0$, the two roots will be *both real* or *both imaginary,* but *opposite* in quality and *arithmetically* equal.

(v) If $c = 0$, one root will be zero and the other $-b/a$.

(vi) If $b = c = 0$, both roots will be zero.

(vii) Both roots will be real, both imaginary, or both complex.

(viii) If $b^2 - 4ac$ is a perfect square, the two roots will be rational when a and b are rational.

The pupil should give the reasons for each of the above statements.

Ex. 1. What kind of numbers are the roots of the equation

$$3x^2 - 2x = -7? \qquad (1)$$

Here $a = 3,\ b = -2,\ c = 7$;

$$\therefore\ b^2 - 4ac = (-2)^2 - 4 \cdot 3 \cdot 7 < 0.$$

Hence the roots of (1) are complex and unequal.

295. Sum and product of roots of $x^2 + b'x + c' = 0$. (A')

Representing the two roots of (A') by x_1 and x_2, we have

$$x_1 = -b'/2 + \sqrt{(b'/2)^2 - c}, \qquad (1)$$

$$x_2 = -b'/2 - \sqrt{(b'/2)^2 - c}. \qquad (2)$$

Adding (1) and (2) to find the sum, we obtain

$$x_1 + x_2 = -b'. \qquad (3)$$

Multiplying (1) by (2) to find the product, we obtain

$$x_1 \cdot x_2 = (-b'/2)^2 - [(b'/2)^2 - c] = c. \qquad (4)$$

Hence, *if a quadratic equation is in the form*

$$x^2 + b'x + c' = 0, \qquad (A')$$

the sum of its roots is equal to minus the coefficient of ***x****, and the product of its roots is equal to the known term.*

E.g., the equation $3x^2 = 7x + 5$ put in the form of (A') becomes

$$x^2 - \tfrac{7}{3}x - \tfrac{5}{3} = 0.$$

Hence the sum of the roots is $7/3$, and the product is $-5/3$.

Note that this principle agrees with § 139 in factoring, and that it is in reality only another form of stating the principle in that article.

Exercise 106.

1. By § 294, what kind of numbers are the roots of each of the equations from 5 to 14 in exercise 104?

2. By § 295, what is the sum and what the product of the roots of each of the equations from 7 to 18 in exercise 105?

Solve each of the following fractional equations:

3. $\frac{5x+7}{x-1}=3x+2.$

4. $\frac{5x-1}{x+1}=\frac{3x}{2}.$

5. $\frac{3x-8}{x-2}=\frac{5x-2}{x+5}.$

6. $\frac{5x-7}{7x-5}=\frac{x-5}{2x-13}.$

7. $\frac{4}{x-1}-\frac{5}{x+2}=\frac{3}{x}.$

8. $\frac{5}{x-2}-\frac{4}{x}=\frac{3}{x+6}.$

9. $\frac{x}{x-1}+\frac{x-1}{x}=\frac{13}{6}.$

10. $\frac{x+1}{x+2}+\frac{x+2}{x+1}=\frac{29}{10}.$

11. $\frac{x}{x-2}+\frac{4}{x+1}=3.$

12. $\frac{1}{2}+\frac{1}{3+x}+\frac{1}{2+3x}=0.$

13. $\frac{5}{x+1}-\frac{10}{x+10}=\frac{2}{3x-3}.$

14. $\frac{1}{3-x}-\frac{4}{5}=\frac{1}{9-2x}.$

15. $\frac{1}{1+x}-\frac{1}{3-x}=\frac{6}{35}.$

16. $\frac{x+4}{x-4}+\frac{x-2}{x-3}=6\frac{1}{3}.$

17. $3\frac{x-1}{x+1}-2\frac{x+1}{x-1}=5.$

18. $\frac{1}{3}+\frac{1}{x+3}+\frac{1}{x+11}=0.$

19. $\frac{3}{x-1}+\frac{4}{x-3}=\frac{15}{x+3}.$

20. $\frac{2x-3}{4}=\frac{x-5}{12}+\frac{5x-16}{x-1}.$

21. $\frac{2x-2}{2x-3}+\frac{3-3x}{3x-2}=\frac{5}{8x-12}.$

22. $\frac{5x}{x-3}-\frac{6}{x+2}+\frac{19}{3-x}=0.$

23. $\frac{2x-1}{2x+1}+\frac{13}{11}=\frac{3x+5}{3x-5}.$

24. $\frac{3x}{x-2}-\frac{4}{x+3}+\frac{4}{2-x}=0.$

25. $\dfrac{x-1}{x+1}+\dfrac{x+1}{x-1}=\dfrac{5x}{x^2-1}.$

26. $\dfrac{1}{x^2-3x}-\dfrac{1}{9-x^2}=\dfrac{13}{16x}.$

27. $\dfrac{1}{x^2-1}-\dfrac{1}{1-x}=\dfrac{7}{8}-\dfrac{1}{x+1}.$

28. $\dfrac{1}{x^2-3x}+\dfrac{1}{x^2+4x}=\dfrac{9}{2x^2}.$

29. $\dfrac{1}{3x-6}+\dfrac{7x}{72(x+2)}=\dfrac{5}{x^2-4}.$

30. $\dfrac{1}{x^2-4}-\dfrac{3}{2-x}=1+\dfrac{1}{3x+6}.$

31. $\dfrac{x}{x+1}+\dfrac{x+1}{x+2}=\dfrac{x-2}{x-1}+\dfrac{x-1}{x}.$

32. $\dfrac{1}{x}+\dfrac{1}{x+4}=\dfrac{1}{x+1}+\dfrac{1}{x+2}.$

33. $\dfrac{x}{x-3}-\dfrac{x-3}{x}+\dfrac{x}{x+3}-\dfrac{x+3}{x}=\dfrac{2}{3}.$

34. $x+\dfrac{1}{a}=a+\dfrac{1}{x}.$

35. $\dfrac{a}{a+x}+\dfrac{a}{a-x}=4.$

36. $\dfrac{1}{x-a}+\dfrac{1}{x-b}=\dfrac{1}{a}+\dfrac{1}{b}.$

37. $\dfrac{a}{x-a}+\dfrac{b}{x-b}=\dfrac{a}{b}+\dfrac{b}{a}.$

38. $\dfrac{x}{a}+\dfrac{a}{x}=\dfrac{b}{a}+\dfrac{a}{b}.$

39. $\dfrac{1}{x-a-b}=\dfrac{1}{x}-\dfrac{1}{a}-\dfrac{1}{b}.$

40. $\dfrac{a^2}{x-b}+\dfrac{b^2}{x-a}=a+b.$

41. $\dfrac{1}{x+a+b}=\dfrac{1}{x}+\dfrac{1}{a}+\dfrac{1}{b}.$

42. $\dfrac{1}{x+a}+\dfrac{1}{x+b}=\dfrac{1}{c+a}+\dfrac{1}{c+b}.$

43. $\dfrac{x}{x+a}+\dfrac{x}{x+b}=\dfrac{c}{c+a}+\dfrac{c}{c+b}.$

44. $\dfrac{x+a}{x-a}+\dfrac{x+b}{x-b}=\dfrac{x-a}{x+a}+\dfrac{x-b}{x+b}.$

45. $\dfrac{a}{x+a}+\dfrac{b}{x+b}=\dfrac{a-c}{x+a-c}+\dfrac{b+c}{x+b+c}.$

46. $\dfrac{1}{x+a+\dfrac{1}{x+b}}=\dfrac{1}{x-a+\dfrac{1}{x-b}}.$

47. $\dfrac{a+b}{x+b}+\dfrac{a+c}{x+c}=\dfrac{2(a+b+c)}{x+b+c}.$

296. The following examples illustrate how any quadratic expression can be factored by writing it as the difference of two squares.

Ex. 1.
$$\begin{aligned} x^2+4x+9 &\equiv x^2+4x+4-(-5)\\ &\equiv (x+2)^2-(\sqrt{-5})^2\\ &\equiv (x+2+\sqrt{-5})(x+2-\sqrt{-5}).\end{aligned}$$

Ex. 2.
$$\begin{aligned} 3x^2+2x-\tfrac{44}{3} &\equiv [(3x)^2+2(3x)-44]\div 3\\ &\equiv [(3x+1)^2-45]\div 3\\ &\equiv (3x+1+3\sqrt{5})(3x+1-3\sqrt{5})\div 3\\ &\equiv (x+\tfrac{1}{3}+\sqrt{5})(3x+1-3\sqrt{5}).\end{aligned}$$

Exercise 107.

Factor each of the quadratic expressions:

1. x^2+6x+7.
2. x^2+8x+5.
3. $x^2-10x+31$.
4. $9x^2-6x-26$.
5. $3x^2+6x-3$.
6. $x^2-14x+52$.
7. $x^2+10x+40$.
8. $x^2-8x+32$.
9. $x^2-\frac{2}{3}x-\frac{4}{3}$.
10. $x^2+\frac{2}{5}x-3$.
11. $x^2-\frac{4}{3}x+5$.
12. $3x^2-8x+7$.

13. Put each of the twelve foregoing trinomials equal to 0, and determine, (1) the sum and the product of the roots of each resulting equation, (2) the character of the roots as real, imaginary, or complex.

Factor each of the following expressions and then find the roots of the equation formed by putting it equal to 0:

14. $x^2 - 4x + 16.$

15. $x^2 - 6x + 11.$

16. $x^2 - 8x + 20.$

17. $4x^2 + 8x + 10.$

18. $9x^2 + 18x + 18.$

19. $16x^2 + 32x + 27.$

CHAPTER XX

PROBLEMS

297. The *solving* of a problem by equations consists of three distinct parts:

(i) The **statement** of the conditions of the problem by one or more equations.

(ii) The **solving** of these equations.

(iii) The **discussion.** A problem may require for an answer a whole number, an arithmetic number, a real number, or numbers having some relation that is not expressed by the equations.

To state these and other such conditions of a problem, and to determine what solutions of the equations give answers to the problem, is called the *discussion* of the problem.

Prob. 1. Eleven times the number of persons in a room is equal to twice the square of that number increased by 12. How many persons are in the room?

Statement. Let $x =$ the *number* of persons;

then $$11x = 2x^2 + 12. \quad (1)$$

Solving (1), we obtain $x = 4,\ x = 3/2.$ (2)

Equations (2) are together equivalent to (1).

Discussion. The number of persons must be an *arithmetic whole* number which satisfies one of the equations in (2); but 4 is the only such number. Hence the one, and only, answer is 4 persons.

Prob. 2. A train travels 300 miles at a uniform rate; if the rate had been 5 miles an hour more, the journey would have taken 2 hours less. Find the rate of the train.

Statement. Let $x =$ the *number* of miles travelled per hour;

then $300 \div x =$ the number of hours required for the journey,

and $300 \div (x + 5) =$ the number of hours the journey would have taken if the rate had been increased 5 miles an hour.

Hence, by the conditions of the problem, we have

$$\frac{300}{x} = \frac{300}{x+5} + 2. \qquad (1)$$

Solving (1), we obtain $x = 25,\ x = -30.$ (2)

Discussion. The number of miles per hour must be an *arithmetic* number which satisfies one of the equations in (2); but 25 is the only such number. Hence the one and only answer is 25 miles an hour.

Prob. 3. The square of the number of dollars a man is worth exceeds by 300 twenty times that number. How much is the man worth?

Statement. Let $x =$ the number of dollars the man is worth;

then $$x^2 = 20x + 300. \qquad (1)$$

Solving (1), we obtain $x = 30,\ x = -10.$

Discussion. If a debt is regarded as a negative possession, both of these roots give answers; that is, the man either has \$30 or owes \$10.

Prob. 4. The sum of the ages of a father and son is 100 years; and one-tenth of the product of their ages, in years, exceeds the father's age by 180. How old is each?

Statement. Let $x =$ the number of years in the father's age;

then $100 - x =$ the number of years in the son's age.

Hence $$0.1\,x(100 - x) = x + 180. \qquad (1)$$

Solving (1), we obtain $x = 60$, and $100 - x = 40$,

or $x = 30$, and $100 - x = 70$.

Discussion. The father must be older than the son; hence the father must be 60, and the son 40, years old.

Both of the solutions of (1) would give answers if the problem read as follows: The sum of the ages of *two persons* is 100 years; and one-tenth of the product of their ages, in years, exceeds the age of one of them by 180. How old is each?

Prob. 5. Find a *real* number whose square increased by 13 is equal to 4 times the number.

Statement. Let $x =$ the number;

then
$$x^2 + 13 = 4x. \tag{1}$$

Solving (1), we obtain
$$x = 2 \pm 3\sqrt{-1}. \tag{2}$$

Discussion. Since there is no *real* number which satisfies (2), the problem is impossible.

If the word "*real*" were omitted in the problem, both the values of x in (2) would be answers.

Prob. 6. A cistern can be filled by two pipes running together in $22\frac{1}{2}$ minutes; the larger pipe alone would fill the cistern in 24 minutes less than the smaller one. Find in what time each would fill it.

Statement. Suppose the larger pipe to fill the cistern in x minutes; then the smaller pipe will fill it in $x + 24$ minutes. Also, $1/x$ and $1/(x + 24)$ are the portions of the cistern which each pipe will fill in one minute, and $1/22\frac{1}{2}$ is the portion that both together will fill in one minute.

Hence
$$\frac{1}{x} + \frac{1}{x + 24} = \frac{1}{22\frac{1}{2}}. \tag{1}$$

Solving (1), we obtain
$$x = 36,\ x = -15. \tag{2}$$

Discussion. The answer must be an arithmetic number, but 36 is the only such number which will satisfy either equation in (2).

Hence the larger pipe would fill the cistern in 36 minutes, and the smaller one in $36 + 24$, or 60, minutes.

Exercise 108.

1. Find two arithmetic numbers one of which is 4 times the other, and whose product is 196.

2. Find two arithmetic numbers whose sum is 25, and whose product is 144.

3. Find two numbers whose sum is 15, and whose product is -250.

4. Divide 71 into two parts, the sum of the squares of which is 2561.

5. A rectangular court is 5 yards longer than it is broad; its area is 1886 square yards. Find its length and breadth.

6. The sum of the squares of two consecutive whole numbers is 1013. Find the numbers.

7. The sum of the reciprocals of two consecutive whole numbers is $\frac{15}{56}$. Find the numbers.

8. If a train travelled 5 miles an hour faster, it would take 1 hour less to travel 210 miles. Find the rate of the train.

9. The perimeter of a rectangular field is 500 yards, and its area is 14,400 square yards. Find the length of the sides.

10. The perimeter of one square exceeds that of another by 100 feet; and the area of the larger square exceeds 3 times the area of the smaller by 325 square feet. Find the length of their sides.

11. A lawn 50 feet long and 34 feet broad has a path of uniform width round it; the area of the path is 540 square feet. Find its width.

12. A man travels 108 miles, and finds that he could have made the journey in $4\frac{1}{2}$ hours less had he travelled 2 miles an hour faster. At what rate did he travel?

13. The product of the sum and difference of an arithmetic number and its reciprocal is $3\frac{3}{4}$. Find the number.

14. A cistern can be filled by 2 pipes in $33\frac{1}{3}$ minutes. To fill the cistern, the larger pipe takes 15 minutes less than the smaller. Find in what time it will be filled by each pipe singly.

15. A hall can be paved with 200 square tiles of a certain size; if each tile were one inch longer each way, it would take 128 tiles. Find the size of the tile.

16. There are two square buildings paved with stones each a foot square. The side of one building exceeds that of the other by 12 feet, and the two pavements together contain 2120 stones. Find the sides of the buildings.

17. Find the number such that the product of the numbers obtained by adding to it 3 and 5 respectively is less by 1 than the square of its double.

18. The plate of a mirror is 18 inches by 12, and it is to be framed with a frame of uniform width, whose area is to be equal to that of the glass. Find the width of the frame.

19. A and B distribute $ 100 each in charity; A relieves 5 persons more than B, and B gives to each $ 1 more than A. How many did they each relieve?

20. The difference between the hypotenuse and two sides of a right-angled triangle is 3 and 6 respectively. Find the sides.

21. In the centre of a square garden is a square lawn; outside this is a gravel walk 4 feet wide, and then a flower border 6 feet wide. If the flower border and lawn together contain 721 square feet, find the area of the lawn.

22. What is the property of a person whose income is $ 2150, when he has $\frac{2}{3}$ of it invested at 4 per cent, $\frac{1}{4}$ at 3 per cent, and the remainder at 2 per cent?

23. A person bought a certain number of oxen for $ 1200, and, after losing 3, sold the rest for $ 20 a head more than they cost him, thus gaining $ 35 by the bargain. How many oxen did he buy?

24. A can do a piece of work in 10 days; but after he has been upon it 4 days, B is sent to help him, and they finish it together in 2 days. In what time would B have done the whole work?

25. A and B can reap a field together in 7 days, which A alone could reap in 10 days. In what time could B alone reap it?

26. A can build a wall in 8 days, which A and B can do together in 5 days. How long would B take to do it alone?

27. A does $\frac{5}{9}$ of a piece of work in 10 days, when B comes to help him, and they take 3 days more to finish it. How long would B take to do it alone?

28. The tens' digit of a certain number exceeds the units' digit by 4, and when the number is divided by the sum of the digits, the quotient is 7. Find the number.

29. Find a number of three digits, each greater by 1 than that which follows it, so that its excess above $\frac{1}{4}$ of the number formed by inverting the digits shall be 36 times the sum of the digits.

30. A detachment from an army was marching in regular column, with 5 men more in depth than in front; but on the enemy coming in sight, the front was increased by 845 men, and the whole was thus drawn up in 5 lines. Find the number of men.

31. The sum of two numbers is 14, and the quotient of the less divided by the greater is $\frac{9}{16}$ of the quotient of the greater divided by the less.

32. Find two fractions whose sum is $\frac{5}{6}$, and whose difference is equal to their product.

33. Two men start at the same time to meet each other from towns which are 25 miles apart. One takes 18 minutes longer than the other to walk a mile, and they meet in 5 hours. How fast does each walk?

Let $x =$ the number of minutes it takes the first man to walk a mile.

34. A and B together can do a piece of work in a certain time. If they each did one-half of the work separately, A

would have to work one day less, and B 2 days more than before. Find the time in which A and B together do the work.

35. A man bought a certain number of railway shares for \$ 1875; he sold all but 15 of them for \$ 1740, gaining \$ 4 per share on their cost price. How many shares did he buy?

36. The denominator of a fraction exceeds the numerator by 4; and if 5 is taken from each, the sum of the reciprocal of the new fraction and 4 times the original fraction is 5. Find the original fraction.

37. A person swimming in a stream which runs $1\frac{1}{2}$ miles per hour finds that it takes him 4 times as long to swim a mile up the stream as it does to swim the same distance down. At what rate does he swim?

38. What is the property of a person whose income is \$ 1140, when one-twelfth of it is invested at 2 per cent, one-half at 3 per cent, one-third at $4\frac{1}{2}$ per cent, and the remainder pays him no dividend?

39. A person having 7 miles to walk increases his speed one mile an hour after the first mile, and is half an hour less on the road than he would have been had he not altered his rate. How long did it take to walk the 7 miles?

Let x miles an hour be his rate at first.

40. The diagonal and the longer side of a rectangle are together 5 times the shorter side, and the longer side exceeds the shorter by 35 yards. Find the area of the rectangle.

41. The price of photographs is raised 50 cents per dozen; and, in consequence, 4 less than before are sold for \$ 5. Find the original price.

42. A boat's crew can row 8 miles an hour in still water. What is the speed of a river's current if it takes them 2

hours and 40 minutes to row 8 miles up and 8 miles down the river?

Let $x =$ the number of miles the current runs in an hour;

then $$\frac{8}{8+x}+\frac{8}{8-x}=\frac{8}{3}.$$

43. At a concert \$300 was received for reserved seats, and the same amount for unreserved seats. A reserved seat cost 75 cents more than an unreserved seat, but 360 more tickets were sold for unreserved than for reserved seats. How many tickets were sold all together?

44. Out of a cask containing 60 gallons of alcohol a certain quantity is drawn off and replaced by water. Of the mixture a second quantity, 14 gallons more than the first, is drawn off and replaced by water. The cask then contains as much water as alcohol. How much was drawn off the first time?

Let $x =$ the number of gallons drawn off the first time; then, in the first mixture,

$60 - x =$ the number of gallons of alcohol,

and $x =$ the number of gallons of water.

$$\therefore \frac{60-x}{60}(60-x-14)=\frac{x}{60}(60-x-14)+x+14.$$

45. A cyclist rode 180 miles at a uniform rate. If he had ridden 3 miles an hour slower than he did, it would have taken him 3 hours longer. How many miles an hour did he ride?

46. A man drives to a certain place at the rate of 8 miles an hour. Returning by a road 3 miles longer at the rate of 9 miles an hour, he takes $7\frac{1}{2}$ minutes longer than in going. How long is each road?

47. A father's age is equal to the united ages of his 5 children, and 5 years ago his age was double their united ages. How old is the father?

48. A and B are two stations 300 miles apart. Two trains start simultaneously from A and B, each to the opposite station. The train from A reaches B 9 hours, the train from B reaches A 4 hours, after they met. When did they meet, and what was the rate of each train?

49. If a carriage wheel $14\frac{2}{3}$ feet in circumference takes one second more to revolve, the rate of the carriage per hour will be $2\frac{2}{3}$ miles less. How fast is the carriage travelling?

Let $x =$ number of miles travelled per hour; then

$$\frac{1}{x}+\frac{1}{10}=\frac{1}{x-2\frac{2}{3}}.$$

50. The number of square inches in the surface of a cubical block exceeds the number of inches in the sum of its edges by 288. Find its edge and volume.

51. A cistern can be filled by 2 pipes running together in 2 hours 55 minutes. The larger pipe by itself will fill it sooner than the smaller one by 2 hours. Find the time in which each pipe separately will fill it.

52. My gross income is \$ 3000. After paying the income tax, and then deducting from the remainder a percentage less by 1 than that of the income tax, the income is reduced to \$ 2736. Find the rate per cent of the income tax.

53. A set out from C toward D at the rate of 5 miles an hour. After he had gone 45 miles, B set out from D toward C, and went every hour $\frac{1}{23}$ of the entire distance. After travelling as many hours as he went miles in an hour, he met A. Find the distance from C to D.

CHAPTER XXI

IRRATIONAL EQUATIONS

298. An **irrational equation** is an equation one or both of whose members is irrational in an unknown.

In this chapter, as heretofore, the radical sign will denote only the principal root of a number or expression.

E.g., $\sqrt{x^2-2}=x-7$ is an irrational equation, and $\sqrt{x^2-2}$ denotes only the principal square root. Note that we cannot speak of the degree of this or any other irrational equation.

In solving irrational equations we use the following principle:

299. *If both members of an irrational equation are raised to the same integral power, the derived equation will have all the roots of the given one and often others in addition.*

Proof. Let
$$A=B \tag{1}$$
be the given irrational equation.

Squaring,
$$A^2=B^2. \tag{2}$$

By §§ 105 and 106, (2) is equivalent to the equation
$$(A-B)(A+B)=0. \tag{3}$$

By § 74, the roots of (3) include those of $A-B=0$, or (1); hence no root is lost by squaring (1).

But the roots of (3) include also those of $A+B=0$, or $A=-B$; hence any root of $A=-B$ which is not a root of $A=B$ must be introduced by squaring (1).

In like manner the principle can be proved for any other positive integral power.

Ex. 1. Solve the equation $x-6=-\sqrt{x-6}$. (1)

Square, $x^2-12x+36=x-6$. (2)

Transpose, $x^2-13x+42=0$.

Factor, $(x-6)(x-7)=0$. (3)

Now (3), or (2), is satisfied when $x=6$ and $x=7$; but (1) is satisfied only when $x=6$. Hence by squaring (1) the root 7 was introduced.

By § 299, the roots of (2) include those of (1) and also those of

$$A=-B, \text{ or } x-6=\sqrt{x-6}. \tag{4}$$

Equation (4) is satisfied both when $x=6$ and when $x=7$.

Hence if it had been required to solve (4), by squaring we would have obtained (2), and no root would have been introduced.

Notice that we cannot say that (2) is equivalent to (1) and (4) jointly (as would be the case, by § 290, were (1) and (4) rational equations); for (2) has only *two* roots, while (1) and (4) together have *three* roots, 6 being a root of each.

Observe that, since we cannot speak of the degree of an irrational equation, we do not know how many roots it has until we have solved it.

Ex. 2. Solve $2-\sqrt{2x+8}+2\sqrt{x+5}=0$. (1)

Our purpose being to obtain a rational equation, it is better before squaring to put the more complex surd in one member by itself, as below.

Transpose, $2+2\sqrt{x+5}=\sqrt{2x+8}$.

Square, $4+8\sqrt{x+5}+4x+20=2x+8$.

Transpose, $x+8=-4\sqrt{x+5}$. (2)

Square, $x^2+16x+64=16x+80$.

Transpose, $x^2-16=0$. (3)

Hence, by § 299, if (1) has any root, it is 4 or -4. But neither $x=4$ nor $x=-4$ satisfies (1); hence (1) has no root, *i.e.*, it is impossible, and therefore both roots of (3) were introduced by squaring (1) and (2).

If we use both the positive and the negative values of $\sqrt{2x+8}$ and $\sqrt{x+5}$, we obtain in addition to (1) the three equations,

$$2-\sqrt{2x+8}-2\sqrt{x+5}=0, \tag{4}$$

$$2+\sqrt{2x+8}-2\sqrt{x+5}=0, \tag{5}$$

$$2+\sqrt{2x+8}+2\sqrt{x+5}=0. \tag{6}$$

By treating (4), (5), or (6) as we did (1), we would obtain (3). Hence the roots of (3) include the roots of each of the four equations, (1), (4), (5), and (6).

By trial we find that -4 is a root of (4), -4 and $+4$ are both roots of (5), and neither -4 nor $+4$ is a root of (6); that is, (6) expresses an impossible condition as well as (1).

Hence in *rationalizing* (1) or (6), *i.e.*, in *deriving the rational equation* (3) from (1) or (6), *two* roots are introduced.

In rationalizing (4), *one* root is introduced.

In rationalizing (5), *no* root is introduced.

Exercise 109.

Solve each of the following irrational equations:

1. $\sqrt{x-5}=3$.
2. $7-\sqrt{x-4}=3$.
3. $\sqrt{5x-1}=2\sqrt{x+3}$.
4. $2\sqrt{3-7x}=3\sqrt{8x-12}$.
5. $\sqrt{9x^2-11x-5}=3x-2$.
6. $\sqrt{4x^2-7x+1}=2x-1\frac{4}{5}$.
7. $\sqrt{x+25}=1+\sqrt{x}$.
8. $\sqrt{x+3}+\sqrt{x}=5$.
9. $\sqrt{8x+33}-3=2\sqrt{2x}$.
10. $10-\sqrt{25+9x}=3\sqrt{x}$.
11. $\sqrt{9x-8}=3\sqrt{x+4}-2$.
12. $\sqrt{x-4}+3=\sqrt{x+11}$.

13. In each of the foregoing examples, from what other irrational equation or equations would we have derived the same rational equation?

14. $\sqrt{8x+17}-\sqrt{2x}=\sqrt{2x+9}$.
15. $\sqrt{3x-11}+\sqrt{3x}=\sqrt{12x-23}$.
16. $\sqrt{12x-5}+\sqrt{3x-1}=\sqrt{27x-2}$.
17. $\sqrt{x+3}+\sqrt{x+8}=\sqrt{4x+21}$.
18. $\sqrt{x+2}+\sqrt{4x+1}=\sqrt{9x+7}$.
19. $\sqrt{x+4ab}=2a+\sqrt{x}$.
20. $\sqrt{x}+\sqrt{4a+x}=2\sqrt{b+x}$.
21. $\sqrt{x-1}+\sqrt{x}=2\div\sqrt{x}$.

22. $\sqrt{x+5}+\sqrt{x}=10\div\sqrt{x}$.

23. $\sqrt{x}-\sqrt{x-8}=2\div\sqrt{x-8}$.

24. $\sqrt{1+x}+\sqrt{x}=2\div\sqrt{1+x}$.

25. $2\sqrt{x}-\sqrt{4x-3}=1\div\sqrt{4x-3}$.

26. $\sqrt{x}-7=1\div(\sqrt{x}+7)$.

27. $\dfrac{x-1}{\sqrt{x}-1}=3+\dfrac{\sqrt{x}+1}{2}$.

Simplify the first member in example 27.

28. $\dfrac{2x-3}{\sqrt{x-2}+1}=2\sqrt{x-2}-1$.

29. $2=\dfrac{\sqrt{2+x}+\sqrt{2-x}}{\sqrt{2+x}-\sqrt{2-x}}$.

30. $\dfrac{1}{1-x}+\dfrac{1}{\sqrt{x}+1}+\dfrac{1}{\sqrt{x}-1}=0$.

In the next seven examples, first reduce the improper fractions to mixed expressions:

31. $\dfrac{\sqrt{x}+3}{\sqrt{x}-2}=\dfrac{3\sqrt{x}-5}{3\sqrt{x}-13}$.

32. $\dfrac{9\sqrt{x}-23}{3\sqrt{x}-8}=\dfrac{6\sqrt{x}-17}{2\sqrt{x}-6}$.

33. $2-\dfrac{\sqrt{x}+3}{\sqrt{x}+2}=\dfrac{\sqrt{x}+9}{\sqrt{x}+7}$.

34. $\dfrac{6\sqrt{x}-7}{\sqrt{6}-1}-5=\dfrac{7\sqrt{x}-26}{7\sqrt{x}-21}$.

35. $\dfrac{2\sqrt{x}-1}{2\sqrt{x}+\frac{4}{3}}=\dfrac{\sqrt{x}-2}{\sqrt{x}-\frac{4}{3}}$.

36. $\dfrac{6\sqrt{x}-21}{3\sqrt{x}-14}=\dfrac{8\sqrt{x}-11}{4\sqrt{x}-13}$.

37. $\dfrac{12\sqrt{x}-11}{4\sqrt{x}-4\frac{2}{3}}=\dfrac{6\sqrt{x}+5}{2\sqrt{x}+\frac{2}{3}}$.

38. $\sqrt{x+5}+\sqrt{x}=\dfrac{10}{\sqrt{x}}$.

39. $\sqrt{(a-x)}+\sqrt{(b-x)}=\sqrt{(a+b-2x)}$.

40. $\sqrt{(ax+b^2)}-\sqrt{(bx+a^2)}=a-b$.

41. $\sqrt{(a+x)}+\sqrt{(b+x)}=\sqrt{(a+b+2x)}$.

42. $\sqrt{(a-x)}+\sqrt{(b-x)}=\sqrt{(2a+2b)}$.

300. Equations in quadratic form. If an equation has only two unknown terms, and if the unknown factor of one of these terms is the square of the unknown factor of the other, the equation is *in quadratic form.*

E.g., since x^2+3x is the square of $\sqrt{x^2+3x}$, the equation

$(x^2+3x)+5\sqrt{x^2+3x}=7$ is *in quadratic form.*

The following examples illustrate how the principles of quadratic equations can be applied to irrational equations which are, or can be put, in quadratic form.

Ex. 1. Solve $2x^2+3x-5\sqrt{2x^2+3x+9}=-3$. (1)

Add 9, $(2x^2+3x+9)-5\sqrt{2x^2+3x+9}=6$. (2)

Since $2x^2+3x+9$ is the square of $\sqrt{2x^2+3x+9}$, equation (2) is *in quadratic form.* Transposing 6 and factoring, we have

$$(\sqrt{2x^2+3x+9}-6)(\sqrt{2x^2+3x+9}+1)=0. \quad (3)$$

The roots of (3) include the roots of

$$\sqrt{2x^2+3x+9}=6, \quad (4)$$

and of

$$\sqrt{2x^2+3x+9}=-1, \quad (5)$$

but no others.

The roots of (4) are 3 and $-4\frac{1}{2}$; while (5) is an impossible equation, since a *principal* square root cannot be a negative number.

What would be the roots of (1), if the sign before the radical were + ?

Ex. 2. Solve $3x^2-7+3\sqrt{3x^2-16x+21}=16x$. (1)

Transposing $16x$ and adding $28-28$, we obtain

$$(3x^2-16x+21)+3\sqrt{3x^2-16x+21}-28=0. \quad (2)$$

Factor, $(\sqrt{3x^2-16x+21}-4)(\sqrt{3x^2-16x+21}+7)=0.$ (3)

The roots of (3) include the roots of

$$\sqrt{3x^2-16x+21}=4, \qquad (4)$$

and of

$$\sqrt{3x^2-16x+21}=-7, \qquad (5)$$

but no others.

The roots of (4) are 5 and 1/3, and (5) is impossible.

What would be the roots of (1), if the sign before the radical were $-$?

If we could not factor (2) by inspection, by § 293 we would have

$$\sqrt{3x^2-16x+21}=-\tfrac{3}{2}\pm\sqrt{\tfrac{9}{4}+28}=+4 \text{ or } -7.$$

Exercise 110.

Solve each of the following irrational equations:

1. $3x^2-4x+\sqrt{3x^2-4x-6}=18.$
2. $x^2-x+4+\sqrt{x^2-x+4}=2.$
3. $x^2+2x-\sqrt{x^2+2x-6}=12.$
4. $1+\frac{1}{5}\sqrt{x^2+x+5}=\dfrac{10}{\sqrt{x^2+x+5}}.$
5. $x^2+\sqrt{4x^2+24x}=24-6x.$
6. $2x^2+6x=1-\sqrt{x^2+3x+1}.$
7. $2(2x-3)(x-4)-\sqrt{2x^2-11x+15}=60.$
8. $\sqrt{4x^2+2x+7}=12x^2+6x-119.$
9. $2x^2-2x-17+2\sqrt{2x^2-3x+7}=x.$
10. $3x(3-x)=11-4\sqrt{x^2-3x+5}.$
11. $2x^2-4x-\sqrt{x^2-2x-3}=9.$

CHAPTER XXII

HIGHER EQUATIONS

301. The following examples illustrate how the principles of quadratic equations are applied to *higher* equations which are, or can be put, *in quadratic form.*

Ex. 1. Solve $(x^2+2x)^2-5(x^2+2x)-14=0.$ (1)

Factor, $(x^2+2x-7)(x^2+2x+2)=0.$ (2)

Equation (2) is equivalent to the two equations

$$x^2+2x-7=0,\quad x^2+2x+2=0,$$

each of which is readily solved.

Ex. 2. Solve $x^4-8x^3+10x^2+24x+5=0.$ (1)

Adding $6x^2-6x^2$ to the first member, we have

$$(x^4-8x^3+16x^2)-6x^2+24x+5=0,$$

or $$(x^2-4x)^2-6(x^2-4x)+5=0. \quad (2)$$

Factor, $(x^2-4x-5)(x^2-4x-1)=0.$ (3)

Equation (3) is equivalent to the two equations

$$x^2-4x-5=0,\quad x^2-4x-1=0,$$

whose roots are $5,\ -1,\ 2\pm\sqrt{5}.$

Ex. 3. Solve $\dfrac{x^2}{x-1}+\dfrac{x-1}{x^2}=\dfrac{17}{4}.$ (1)

Here the second term is the reciprocal of the first.

Putting y for the first term, and therefore the reciprocal of y for the second, (1) becomes

$$y+\frac{1}{y}=\frac{17}{4}.$$

Multiply by $4y$, $\qquad 4y^2 - 17y + 4 = 0.$

Factor, $\qquad (y-4)(4y-1) = 0.$

$$\therefore\ y = 4, \text{ or } 1/4.$$

Hence (1) is equivalent to the two equations

$$\frac{x^2}{x-1} = 4 \text{ and } \frac{x^2}{x-1} = \frac{1}{4}. \qquad (2)$$

The roots of equations (2) are 2, 2, $(1 \pm \sqrt{-15})/8$.

Exercise 111.

Solve the following equations:

1. $x^4 - 5x^2 + 4 = 0.$
2. $x^4 - 10x^2 + 9 = 0.$
3. $x^4 - 7x^2 - 18 = 0.$
4. $(x^2 - 1)/9 + 1/x^2 = 1.$
5. $x^2 + 100/x^2 = 29.$
6. $(x^2 + x)^2 - 22(x^2 + x) = -40.$
7. $(x^2 - x)^2 - 8(x^2 - x) = -12.$
8. $\left(x + \dfrac{1}{x}\right)^2 + 4\left(x + \dfrac{1}{x}\right) = 12.$
9. $2x^2 + 3x + 1 = 30/(2x^2 + 3x).$
10. $x^2 + 3x - 20/(x^2 + 3x) = 8.$
11. $x^2 + x + 1 = 42/(x^2 + x).$
12. $x^4 - 8x^3 - 12x^2 + 112x = 128.$
13. $x^4 + 2x^3 - 3x^2 - 4x - 96 = 0.$
14. $x^4 - 10x^3 + 30x^2 - 25x + 4 = 0.$
15. $x^4 - 14x^3 + 61x^2 - 84x + 20 = 0.$
16. $\dfrac{x^2}{x+1} + \dfrac{x+1}{x^2} = 2.$
17. $\dfrac{x}{x^2+1} + \dfrac{x^2+1}{x} = \dfrac{5}{2}.$
18. $\dfrac{x^2+2}{x^2+4x+1} + \dfrac{x^2+4x+1}{x^2+2} = \dfrac{5}{2}.$

302. A **binomial equation** is an equation of the form $x^n = a$, where n is a positive integer.

The binomial quadratic equation $x^2 = a$ has already been solved. Certain binomial higher equations are readily solved by previous principles.

Ex. 1. Solve the binomial cubic equation $x^3 - 1 = 0$. (1)

Factor, $$(x-1)(x^2+x+1)=0. \quad (2)$$

Equation (2) is equivalent to the two equations

$$x - 1 = 0,\ x^2 + x + 1 = 0. \quad (3)$$

The solutions of equations (3) are 1 and $(-1 \pm \sqrt{-3})/2$.

Hence, the cubic equation (1) has one real and two complex solutions.

Since by (1), $x^3 = {}^+1$, the cube of each solution of (1) is equal to $^+1$; that is, $^+1$ has the three cube roots $^+1$ $(-1+\sqrt{-3})/2$, and $(-1-\sqrt{-3})/2$. See example 32, exercise 103.

Since $+27 = (+1) \times 27$, the three cube roots of $^+27$, or the three solutions of the cubic equation $x^3 = {}^+27$, can be obtained by multiplying the three cube roots of $^+1$ by the cube root of the arithmetic number 27.

Thus the three solutions of $x^3 = {}^+27$ are $^+3$ and $3(-1 \pm \sqrt{-3})/2$.

Ex. 2. Solve the binomial biquadratic equation $x^4 - 1 = 0$. (1)

Factor, $$(x^2-1)(x^2+1)=0. \quad (2)$$

The solutions of (2) are ± 1 and $\pm\sqrt{-1}$.

Hence $^+1$ has four fourth roots, two real and two imaginary.

The four solutions of $x^4 = 81$, or the four fourth roots of $^+81$, are ± 3 and $\pm 3\sqrt{-1}$.

Ex. 3. Solve $$x^5 = 1, \text{ or } x^5 - 1 = 0. \quad (1)$$

Factor, $$(x-1)(x^4+x^3+x^2+x+1)=0. \quad (2)$$

One solution of (2) is 1, and the other solutions are those of the equation

$$x^4 + x^3 + x^2 + x + 1 = 0. \quad (3)$$

Divide by x^2, $$x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2} = 0.$$

Add 1, $$x^2+2+\frac{1}{x^2}+x+\frac{1}{x}=1,$$

or $$\left(x+\frac{1}{x}\right)^2+\left(x+\frac{1}{x}\right)=1.$$

$$\therefore\ x+\frac{1}{x}=-\tfrac{1}{2}\pm\sqrt{1+\tfrac{1}{4}}=-\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{5}.$$

$$\therefore\ x^2+1=\tfrac{1}{2}(-1\pm\sqrt{5})x. \qquad (4)$$

Solving the two equations in (4), we obtain four solutions, all of which are complex. Hence 1, or any other positive number, has five fifth roots, one real and four complex.

Ex. 4. Solve $$x^6=1,\ \text{or}\ x^6-1=0. \qquad (1)$$

Factor, $$(x^3-1)(x^3+1)=0,$$

or $$(x-1)(x^2+x+1)(x+1)(x^2-x+1)=0. \qquad (2)$$

Equation (2) is equivalent to the four equations

$$x-1=0,\ x^2+x+1=0,\ x+1=0,\ x^2-x+1=0. \qquad (3)$$

Solving equations (3), we obtain six solutions, two real and four complex.

Hence 1, or any other *positive* number, has six sixth roots, two real and four complex.

Ex. 5. Solve $$x^8=1,\ \text{or}\ x^8-1=0. \qquad (1)$$

Factor, $$(x^4+1)(x^2+1)(x^2-1)=0. \qquad (2)$$

The roots of (2) are ± 1, $\pm\sqrt{-1}$, and the roots of

$$x^4+1=0. \qquad (3)$$

Add $2x^2-2x^2$, $$x^4+2x^2+1-2x^2=0.$$

Factor, $$(x^2+1+x\sqrt{2})(x^2+1-x\sqrt{2})=0. \qquad (4)$$

Equation (4) is equivalent to the two equations

$$x^2+1+x\sqrt{2}=0,$$

and $$x^2+1-x\sqrt{2}=0,$$

each of which has two complex roots.

Hence, any *positive* number has eight eighth roots, two real, two imaginary, and four complex.

Observe that any root of $+1$ or -1 is a quality-unit.

Exercise 112.

Solve each of the following binomial equations:

1. $x^3 + 1 = 0$.
2. $x^3 + 27 = 0$.
3. $x^4 + 1 = 0$.
4. $x^4 + 16 = 0$.
5. $x^5 + 1 = 0$.
6. $x^5 + 32 = 0$.
7. $x^6 + 1 = 0$.
8. $x^6 + 64 = 0$.
9. $x^3 - 64 = 0$.
10. $x^4 - 625 = 0$.
11. $x^5 - 243 = 0$.
12. $x^6 - 729 = 0$.

CHAPTER XXIII

SYSTEMS INVOLVING QUADRATIC AND HIGHER EQUATIONS

303. As in linear systems, so in any other determinate system there must be as many independent consistent equations as there are unknowns.

In solving systems which involve quadratic or higher equations we have frequent use for the following principle of equivalent systems:

304. *If M, N, P, Q denote any integral unknown expressions, then system* (a)

$$\left.\begin{aligned} M \times N = 0, \\ P \times Q = 0, \end{aligned}\right\} \qquad (a)$$

is equivalent to the four systems (b), (c), (d), (e).

$$\left.\begin{aligned} M = 0, \\ P = 0. \end{aligned}\right\} (b) \qquad \left.\begin{aligned} M = 0, \\ Q = 0. \end{aligned}\right\} (c) \qquad \left.\begin{aligned} N = 0, \\ P = 0. \end{aligned}\right\} (d) \qquad \left.\begin{aligned} N = 0, \\ Q = 0. \end{aligned}\right\} (e)$$

Proof. Any solution of system (a) must reduce the factor M or N (or both) to 0, and at the same time must reduce P or Q (or both) to 0.

Now any solution of system (a) which reduces M to 0 and P to 0 is a solution of system (b); any solution of (a) which reduces M to 0 and Q to 0 is a solution of (c); and so on. Hence any solution of system (a) is a solution of system (b), (c), (d), or (e).

Conversely, any solution of system (b) reduces M to 0 and P to 0, and therefore reduces $M \times N$ to 0 and $P \times Q$ to 0;

hence, any solution of system (b) is a solution of system (a), and so on. Hence, any solution of system (b), (c), (d), or (e) is a solution of system (a).

Whence system (a) is equivalent to the four systems (b), (c), (d), (e).

Ex. 1. Solve the system

$$\left.\begin{aligned} x^2 - xy - 2y^2 &= 0, & (1)\\ 3y^2 - 10y + 8 &= 0. & (2)\end{aligned}\right\}(a)$$

By § 201, system (a) is equivalent to (b).

$$\left.\begin{aligned} &\text{Factor (1),} & (x-2y)(x+y) &= 0. & (3)\\ &\text{Factor (2),} & (3y-4)(y-2) &= 0. & (4)\end{aligned}\right\}(b)$$

By § 304, (b) is equivalent to the four linear systems (c).

$$\left.\begin{aligned} x-2y&=0,\\ 3y-4&=0.\end{aligned}\right\}\quad \left.\begin{aligned} x-2y&=0,\\ y-2&=0.\end{aligned}\right\}\quad \left.\begin{aligned} x+y&=0,\\ 3y-4&=0.\end{aligned}\right\}\quad \left.\begin{aligned} x+y&=0,\\ y-2&=0.\end{aligned}\right\}\quad (c)$$

The solutions of the four systems (c) are $\frac{8}{3}, \frac{4}{3}$; $4, 2$; $-\frac{4}{3}, \frac{4}{3}$; $-2, 2$; which are therefore the four solutions of (a).

Ex. 2. Solve the system

$$\left.\begin{aligned} x^2 + 2xy + y^2 &= 36, & (1)\\ x^2 - 2xy &= 0. & (2)\end{aligned}\right\}(a)$$

System (a) is equivalent to system (b).

$$\left.\begin{aligned} &\text{From (1),} & x+y &= \pm 4. & (3)\\ &\text{From (2),} & x(x-2y) &= 0. & (4)\end{aligned}\right\}(b)$$

By § 304, (b) is equivalent to the four linear systems (c).

$$\left.\begin{aligned} x+y&=4,\\ x&=0.\end{aligned}\right\}\quad \left.\begin{aligned} x+y&=4,\\ x-2y&=0.\end{aligned}\right\}\quad \left.\begin{aligned} x+y&=-4,\\ x&=0.\end{aligned}\right\}\quad \left.\begin{aligned} x+y&=-4,\\ x-2y&=0.\end{aligned}\right\}\quad (c)$$

In applying the principle of this article to system (b), observe that the two equations in (3) are equivalent to the equation

$$(x+y-4)(x+y+4)=0.$$

The solutions of (a) are therefore $0, 4$; $\frac{8}{3}, \frac{4}{3}$; $0, -4$; $-\frac{8}{3}, -\frac{4}{3}$.

Whenever *one* or *each* of the equations of a system can be resolved into two or more equivalent equations, the *first* step in solving the system is to apply the principle of this article.

305. The two examples in § 304 illustrate the theorem:

A system of two quadratic equations in two unknowns has, in general, four, and only four, solutions.

Exercise 113.

Solve each of the following systems of equations:

1. $\left.\begin{array}{l}(x-2y)(x-1)=0,\\ x+y-4=0.\end{array}\right\}$

2. $\left.\begin{array}{l}(x-3)(y-2)=0,\\ x+y=7.\end{array}\right\}$

3. $\left.\begin{array}{l}x^2-4xy+3y^2=0,\\ x+y=1.\end{array}\right\}$

4. $\left.\begin{array}{l}xy-7y+3x=21,\\ x+y=2.\end{array}\right\}$

5. $\left.\begin{array}{l}4x^2-xy=0,\\ 2x-3y=6.\end{array}\right\}$

6. $\left.\begin{array}{l}(x+y)(x-y+1)=0,\\ (x+2)(y+3)=0.\end{array}\right\}$

7. $\left.\begin{array}{l}(x+y)^2=16,\\ (x-y)^2=4.\end{array}\right\}$

8. $\left.\begin{array}{l}x^2+2xy+y^2=144,\\ x^2-2xy+y^2=4.\end{array}\right\}$

9. $\left.\begin{array}{l}x^2+xy=x+y,\\ y^2-2xy=3y-6x.\end{array}\right\}$

10. $\left.\begin{array}{l}x^2-y^2=x+y,\\ x^2-3xy=5x-15y.\end{array}\right\}$

306. A system of two equations, *one linear* and the *other quadratic*, can be solved by first *eliminating* one unknown *by substitution*.

Ex. 1. Solve the system

$$\left.\begin{array}{lr}x+2y=5, & (1)\\ x^2+2y^2=9. & (2)\end{array}\right\}(a)$$

$$\left.\begin{array}{llr}\text{Solve (1) for } x, & x=5-2y. & (3)\\ \text{From (2) and (3),} & (5-2y)^2+2y^2=9. & \\ \text{Factor,} & (3y-4)(2y-4)=0. & (4)\end{array}\right\}(b)$$

By § 201, (a) is equivalent to the system, (3) and (4), or (b).

By § 304, (b) is equivalent to the two systems (c) and (d).

$$\left.\begin{aligned} x &= 5 - 2y, \\ 3y - 4 &= 0. \end{aligned}\right\} (c) \qquad \left.\begin{aligned} x &= 5 - 2y, \\ y - 2 &= 0. \end{aligned}\right\} (d)$$

The solution of (c) is $7/3$, $4/3$; and that of (d) is 1, 2.

After the theory is clearly understood, the work after equation (4) can be abridged as below:

From (4), $\quad y = 4/3$, or 2.

When $y = 4/3$, from (3), $x = 5 - 8/3 = 7/3$;

When $y = 2$, from (3), $\quad x = 5 - 4 = 1$.

This example illustrates the following theorem:

A system of one linear and one quadratic equation in two unknowns has, in general, two, and only two, solutions.

Exercise 114.

Solve each of the following systems:

1. $\left.\begin{aligned} x + y &= 15, \\ xy &= 36. \end{aligned}\right\}$

2. $\left.\begin{aligned} x + y &= 51, \\ xy &= 518. \end{aligned}\right\}$

3. $\left.\begin{aligned} 8x - 4y &= -12, \\ 3x^2 + 2y^2 - y &= 48. \end{aligned}\right\}$

4. $\left.\begin{aligned} x - y &= 10, \\ x^2 + y^2 &= 58. \end{aligned}\right\}$

5. $\left.\begin{aligned} 3x + 3y &= 10, \\ xy &= 1. \end{aligned}\right\}$

6. $\left.\begin{aligned} 2x - 5y &= 0, \\ x^2 - 3y^2 &= 13. \end{aligned}\right\}$

7. $\left.\begin{aligned} 2x + 3y &= 0, \\ 4x^2 + 9xy + 9y^2 &= 72. \end{aligned}\right\}$

8. $\left.\begin{aligned} x - y &= 3, \\ x^2 + 19 + y^2 &= 3xy. \end{aligned}\right\}$

9. $\left.\begin{aligned} 2x - y &= 5, \\ x + 3y &= 2xy. \end{aligned}\right\}$

10. $\left.\begin{aligned} 3x + 2y &= 5, \\ x^2 - 4xy + 5y^2 &= 2. \end{aligned}\right\}$

11. $\left.\begin{aligned} 3x^2 - 2xy &= 15, \\ 2x + 3y &= 12. \end{aligned}\right\}$

12. $\left.\begin{aligned} x + y &= 15, \\ x^2 + y^2 &= 125. \end{aligned}\right\}$

13. $\left.\begin{aligned} x^2 + 3xy - y^2 &= 23, \\ x + 2y &= 7. \end{aligned}\right\}$

14. $\left.\begin{aligned} x^2 + y^2 &= 185, \\ x - y &= 3. \end{aligned}\right\}$

15. $\left.\begin{aligned} 2x - 7y &= 25, \\ 5x^2 + 4xy + 3y^2 &= 23. \end{aligned}\right\}$

16. $\left.\begin{aligned} 3x - 31 &= 5y, \\ x^2 + 5xy + 25 &= y^2. \end{aligned}\right\}$

17. $\left.\begin{aligned} x + y &= 2, \\ 2x + 3y &= 6xy. \end{aligned}\right\}$

18. $\left.\begin{aligned} x + 2y &= 7, \\ 3y + 6x &= 5xy. \end{aligned}\right\}$

19. $\left.\begin{aligned} x - y &= 1, \\ x^2 - y^2 &= (5/6)xy. \end{aligned}\right\}$

20. $\left.\begin{aligned} x^2 - 2xy &= 0, && (1) \\ 4x^2 + 9y^2 &= 225. && (2) \end{aligned}\right\} (a)$

Factor (1), $\qquad x(x - 2y) = 0. \qquad (3)$

System, (2) and (3), which by § 201 is equivalent to (a), is equivalent to the two systems (b) and (c).

$$\left.\begin{aligned} 4x^2 + 9y^2 &= 225, \\ x &= 0. \end{aligned}\right\} (b) \qquad \left.\begin{aligned} 4x^2 + 9y^2 &= 225, \\ x - 2y &= 0. \end{aligned}\right\} (c)$$

21. $\left.\begin{aligned} x^2 - 3xy &= 0, \\ 5x^2 + 3y^2 &= 48. \end{aligned}\right\}$

22. $\left.\begin{aligned} 2x^2 - 3xy &= 0, \\ y^2 + 5xy &= 34. \end{aligned}\right\}$

23. $\left.\begin{aligned} x^2 - 2xy + 5 &= 0, \\ (x - y)^2 &= 4. \end{aligned}\right\}$

24. $\left.\begin{aligned} x^2 + 4y^2 &= 4xy + 16, \\ x^2 + y^2 &= 5. \end{aligned}\right\}$

307. If each of two quadratic equations has *one, and only one, term below the 2d degree, and these two terms are similar;* the system can be solved by first *eliminating* the term below the second degree by *addition* or *subtraction.*

Ex. 1. Solve the system $\left.\begin{aligned} x^2 + xy + 2y^2 &= 44, && (1) \\ 2x^2 - xy + y^2 &= 16. && (2) \end{aligned}\right\} (a)$

Each equation in system (a) has one, and only one, term below the 2d degree, 44 and 16, respectively; and these terms are similar. We proceed to eliminate the term below the 2d degree.

Multiply (1) by 4, $\qquad 4x^2 + 4xy + 8y^2 = 176. \qquad (3)$

Multiply (2) by 11, $\qquad 22x^2 - 11xy + 11y^2 = 176. \qquad (4)$

Subtract (3) from (4), $\qquad 18x^2 - 15xy + 3y^2 = 0. \qquad (5)$

Factor, $\qquad (y - 3x)(y - 2x) = 0. \qquad (6)$

System, (6) and (1), which is equivalent to (a), is equivalent also to the two systems (b).

$$\left.\begin{aligned} x^2 + xy + 2y^2 &= 44, \\ y - 3x &= 0. \end{aligned}\right\} \qquad \left.\begin{aligned} x^2 + xy + 2y^2 &= 44, \\ y - 2x &= 0. \end{aligned}\right\} \qquad (b)$$

The solutions of systems (b) are $\sqrt{2},\ 3\sqrt{2}$; $-\sqrt{2},\ -3\sqrt{2}$; $2,\ 4$; and $-2,\ -4$; which are therefore all the solutions of (a).

Ex. **2.** Solve the system

$$\left.\begin{aligned} y^2 - 2x^2 &= 4x, & (1) \\ 3y^2 + xy - 2x^2 &= 16x. & (2) \end{aligned}\right\} (a)$$

The terms below the 2d degree, $4x$ and $16x$, are similar.

We proceed to eliminate the term in x.

Multiply (1) by 4, $\quad 4y^2 - 8x^2 = 16x. \quad (3)$

Subtract (2) from (3), $\quad y^2 - xy - 6x^2 = 0.$

Factor, $\quad (y + 2x)(y - 3x) = 0. \quad (4)$

System, (4) and (1), which is equivalent to system (a), is equivalent also to the two systems (b) and (c).

$$\left.\begin{aligned} y^2 - 2x^2 &= 4x, \\ y + 2x &= 0. \end{aligned}\right\} (b) \qquad \left.\begin{aligned} y^2 - 2x^2 &= 4x, \\ y - 3x &= 0. \end{aligned}\right\} (c)$$

The two solutions of system (b) are $0,\ 0$ and $2,\ -4$; those of (c) are $0,\ 0$ and $4/7,\ 12/7$; which are therefore the four solutions of (a).

Observe that by eliminating the term below the second degree in each of the systems above, we obtained a homogeneous equation in x and y, which we resolved into two equivalent equations.

Instead of eliminating the term below the second degree, it is sometimes better to eliminate one of the terms of the second degree.

Ex. **3.** Solve the system

$$\left.\begin{aligned} 9x^2 - 8y^2 &= 28, & (1) \\ 7x^2 + 3y^2 &= 31. & (2) \end{aligned}\right\} (a)$$

Multiplying (1) by 3 and (2) by 8, and adding, we eliminate y^2 and obtain

$$83x^2 = 332, \text{ or } x = \pm 2. \qquad (3)$$

When $x = 2$, from (2) we obtain $\quad y = \pm 1.$

When $x = -2$, from (2) we obtain $\quad y = \pm 1.$

Hence the four solutions of (a) are 2, 1; 2, -1; -2, 1; -2, -1.

Ex. 4. Solve the system

$$\left.\begin{aligned} xy + x &= 25, &\quad (1)\\ 2xy - 3y &= 28. &\quad (2)\end{aligned}\right\}(a)$$

Eliminating the product xy we obtain

$$2x + 3y = 22. \qquad (3)$$

Solving system, (1) and (3), which is equivalent to system (a), we obtain the two solutions 5, 4; 15/2, 7/3.

Ex. 5. Solve the system

$$\left.\begin{aligned} x^2 - 3xy &= 10, &\quad (1)\\ 4y^2 - xy &= -1. &\quad (2)\end{aligned}\right\}(a)$$

Sometimes by adding or subtracting the given equations we obtain an equation which can be resolved into equivalent equations.

Add (1) and (2), $x^2 - 4xy + 4y^2 = 9$,

or

$$x - 2y = \pm 3. \qquad (3)$$

System, (1) and (3), which is equivalent to system (a), is equivalent to the two systems (b) and (c).

$$\left.\begin{aligned} x^2 - 3xy &= 10,\\ x - 2y &= 3.\end{aligned}\right\}(b) \qquad \left.\begin{aligned} x^2 - 3xy &= 10,\\ x - 2y &= -3.\end{aligned}\right\}(c)$$

Exercise 115.

Solve each of the following systems of equations:

1. $\left.\begin{aligned} x^2 + xy &= 12,\\ xy - y^2 &= 2.\end{aligned}\right\}$

2. $\left.\begin{aligned} x^2 + xy &= 24,\\ 2y^2 + 3xy &= 32.\end{aligned}\right\}$

3. $\left.\begin{aligned} x^2 + 3xy &= 7,\\ y^2 + xy &= 6.\end{aligned}\right\}$

4. $\left.\begin{aligned} 3x^2 - 5y^2 &= 28,\\ 3xy - 4y^2 &= 8.\end{aligned}\right\}$

5. $\left.\begin{aligned} x^2 - 3xy + 2y^2 &= 3,\\ 2x^2 + y^2 &= 6.\end{aligned}\right\}$

6. $\left.\begin{aligned} x^2 + 5y^2 &= 84,\\ 3x^2 + 17xy + 84 &= y^2.\end{aligned}\right\}$

7. $\left.\begin{aligned} x^2 - 7xy - 9y^2 &= 9,\\ x^2 + 5xy + 11y^2 &= 5.\end{aligned}\right\}$

8. $\left.\begin{aligned} x(x + y) &= 40,\\ y(x - y) &= 6.\end{aligned}\right\}$

9. $\left.\begin{aligned} x^2 + xy + y^2 &= 7,\\ 6x^2 - 2xy + y^2 &= 6.\end{aligned}\right\}$

10. $\left.\begin{aligned} x^2 + 3xy &= 28,\\ xy + 4y^2 &= 8.\end{aligned}\right\}$

In example 10, add the two equations.

11. $\left.\begin{array}{l} x^2+3xy=40, \\ 4y^2+xy=9. \end{array}\right\}$

12. $\left.\begin{array}{l} x^2+3xy=54, \\ xy+4y^2=115. \end{array}\right\}$

13. $\left.\begin{array}{l} x^2+xy+44=2y^2, \\ xy+3y^2=80. \end{array}\right\}$

14. $\left.\begin{array}{l} 3xy+x^2=10, \\ 5xy-2x^2=2. \end{array}\right\}$

In example 14, eliminate x^2 or the product xy.

15. $\left.\begin{array}{l} 4x^2-3y^2=-11, \\ 11x^2+5y^2=301. \end{array}\right\}$

16. $\left.\begin{array}{l} 2x^2+y^2=9, \\ 5x^2+6y^2=26. \end{array}\right\}$

17. $\left.\begin{array}{l} 20x^2-16y^2=179, \\ 5x^2-336y^2=24. \end{array}\right\}$

18. $\left.\begin{array}{l} 2x^2-2xy-3y^2=18, \\ 3x^2-2y^2=19. \end{array}\right\}$

19. $\left.\begin{array}{l} x^2+3x-2y=4, \\ 2x^2-5x+3y=-2. \end{array}\right\}$

20. $\left.\begin{array}{l} (x+1)(y+1)=10, \\ xy=3. \end{array}\right\}$

21. $\left.\begin{array}{l} 4x^2-3xy=10, \\ y^2-xy=6. \end{array}\right\}$

22. $\left.\begin{array}{l} x^2-2xy=3y, \\ 2x^2-9y^2=9y. \end{array}\right\}$

23. $\left.\begin{array}{l} 2x^2-xy+y^2=2y, \\ 2x^2+4xy=5y. \end{array}\right\}$

24. $\left.\begin{array}{l} x^3+1=9y, \\ x^2+x=6y. \end{array}\right\}$

308. Systems of symmetrical equations. *A symmetrical equation* is one which is not changed by interchanging its unknowns.

E.g., $x+y=12$, $xy=35$, $x^2+y^2=74$, $x^2 \pm 2xy+y^2=16$ are symmetrical equations. The equations $x-y=2$, $x^3-y^3=4$, $x^4-y^4=16$ are symmetrical *except for sign*.

The methods given below for solving systems of symmetrical equations can usually be employed when the equations are symmetrical *except for sign*.

Ex. 1. Solve the system,

$$\left.\begin{array}{lr} x^2+y^2=74, & (1) \\ xy=35, & (2) \end{array}\right\} (a)$$

Multiply (2) by 2,

$$2xy=70. \quad (3)$$

Add (3) to (1), $x^2 + 2xy + y^2 = 144,$

or $x + y = \pm 12.$ (4)

Subtract (3) from (1), $x - y = \pm 2.$ (5) $\Big\} (b)$

By § 304, system (b) is equivalent to the four systems (c).

$$\left.\begin{matrix} x+y=12, \\ x-y=2. \end{matrix}\right\} \quad \left.\begin{matrix} x+y=12, \\ x-y=-2. \end{matrix}\right\} \quad \left.\begin{matrix} x+y=-12, \\ x-y=2. \end{matrix}\right\} \quad \left.\begin{matrix} x+y=-12, \\ x-y=-2. \end{matrix}\right\} (c)$$

The solutions of systems (c) are 7, 5; 5, 7; -5, -7; -7, -5.

Ex. 2. Solve the system,

$$\left.\begin{aligned} x^2 - xy + y^2 &= 49, && (1) \\ x + y &= 13. && (2) \end{aligned}\right\} (a)$$

Square (2), $x^2 + 2xy + y^2 = 169.$ (3)

Subtract (1) from (3), $3xy = 120,$

or $xy = 40.$ (4)

Subtract (4) from (1), $x - y = \pm 3.$ (5)

System, (2) and (5), is equivalent to the two systems (b).

$$\left.\begin{matrix} x+y=13, \\ x-y=3. \end{matrix}\right\} \qquad \left.\begin{matrix} x+y=13, \\ x-y=-3. \end{matrix}\right\} \qquad (b)$$

The solutions of systems (b) are 8, 5, and 5, 8.

The four solutions of system, (1) and (5), must include the two solutions of (a), since no solution was lost by squaring (2).

Hence the two solutions of (a) must satisfy (2) and also (5).

Therefore the solutions of systems (b) are the two solutions of (a).

Observe that each of the above systems was solved *by first finding the values of* $x + y$ *and* $x - y$.

Ex. 3. Solve the system

$$\left.\begin{aligned} x^4 + y^4 &= 82, && (1) \\ x - y &= 2. && (2) \end{aligned}\right\} (a)$$

Let $x = v + w,$ (3)

and $y = v - w.$ (4)

From (2), (3), and (4), $w = 1.$ (5)

From (1), (3), (4), and (5),

$$(v + 1)^4 + (v - 1)^4 = 82,$$

or $(v^2 + 10)(v^2 - 4) = 0.$

$\therefore v = \pm 2, \text{ or } \pm\sqrt{-10}.$ (6)

(b)

$$\left.\begin{array}{ll}\text{From (3), (5), and (6),} & x = 3,\ -1,\ 1 \pm \sqrt{-10}.\\ \text{From (4), (5), and (6),} & y = 1,\ -3,\ -1 \pm \sqrt{-10}.\end{array}\right\} (c)$$

System (a) with (3) and (4) forms a system equivalent to (b), which is equivalent to (c) with (5) and (6).

Hence the four solutions of (a) are given in (c).

Exercise 116.

Solve each of the following systems of equations by first finding the values of $x + y$ and $x - y$:

1. $\left.\begin{array}{l}x^2 + y^2 = 89,\\ xy = 40.\end{array}\right\}$

2. $\left.\begin{array}{l}x^2 + y^2 = 170,\\ xy = 13.\end{array}\right\}$

3. $\left.\begin{array}{l}x^2 + y^2 = 65,\\ xy = 28.\end{array}\right\}$

4. $\left.\begin{array}{l}x^2 + xy + y^2 = 67,\\ x^2 - xy + y^2 = 39.\end{array}\right\}$

5. $\left.\begin{array}{l}x^2 + 1 + y^2 = 3\,xy,\\ 3\,x^2 - xy + 3\,y^2 = 13.\end{array}\right\}$

6. $\left.\begin{array}{l}x^2 - xy + y^2 = 76,\\ x + y = 14.\end{array}\right\}$

7. $\left.\begin{array}{l}x^2 + xy + y^2 = 61,\\ x + y = 9.\end{array}\right\}$

8. $\left.\begin{array}{l}x^2 - 4\,xy + y^2 = 52,\\ \frac{1}{10}(x - y) = 1.\end{array}\right\}$

9. Solve the systems in examples 1, 2, 4, 5, 8, 12, and 14 in exercise 113, by first finding the values of $x + y$ and $x - y$.

Solve each of the following systems of equations:

10. $\left.\begin{array}{l}x - y = 3,\\ x^2 - 3\,xy + y^2 = -19.\end{array}\right\}$

11. $\left.\begin{array}{l}x^2 - xy + y^2 = 72,\\ x + y = 14.\end{array}\right\}$

12. $\left.\begin{array}{l}x + y = 4,\\ x^4 + y^4 = 82.\end{array}\right\}$

13. $\left.\begin{array}{l}x^4 + y^4 = 272,\\ x - y = 2.\end{array}\right\}$

14. $\left.\begin{array}{l}x - y = 2,\\ x^5 - y^5 = 242.\end{array}\right\}$

15. $\left.\begin{array}{l}x^4 + y^4 = 706,\\ x + y = 8.\end{array}\right\}$

16. Solve system (2) in § 263, and observe that x and y are rational only when $a^2 - b$ is a perfect square.

309. Division. *If the members of one equation* (1) *are divided by the corresponding members of another equation* (2), *and the derived equation* (3) *is integral in the unknowns; then the system* (a) *is equivalent to the two systems* (b) *and* (c).

$$\left.\begin{aligned} AB &= A'B', \quad (1)\\ B &= B'. \quad (2)\end{aligned}\right\}(a) \qquad \left.\begin{aligned} A &= A', \quad (3)\\ B &= B'. \quad (4)\end{aligned}\right\}(b) \qquad \left.\begin{aligned} B &= 0, \quad (5)\\ B' &= 0. \quad (6)\end{aligned}\right\}(c)$$

Observe that (3) is the derived equation, that (4) is the same as (2), and that (5) and (6) are formed by equating to 0 the members of (2).

Proof. Substituting B for B' in (1) we obtain the system (d).

$$\left.\begin{aligned} B(A - A') &= 0,\\ B &= B'.\end{aligned}\right\}(d)$$

By § 202, system (d) is equivalent to system (a).

By § 304, system (d) is equivalent to the two systems (b) and (e).

$$\left.\begin{aligned} B &= 0,\\ B &= B'.\end{aligned}\right\}(e)$$

By substitution (§ 202), system (e) is equivalent to (c).

Hence (a) is equivalent to the two systems (b) and (c).

E.g., dividing (1′) by (2′) we obtain the integral equation (3′);

$$\left.\begin{aligned} y^3 &= x(x + y), \quad (1')\\ y^2 &= x + y. \quad (2')\end{aligned}\right\}(a')$$

Hence system (a') is equivalent to the two systems (b') and (c'),

$$\left.\begin{aligned} y &= x, \quad (3')\\ y^2 &= x + y. \quad (4')\end{aligned}\right\}(b') \qquad \left.\begin{aligned} y^2 &= 0, \quad (5')\\ x + y &= 0. \quad (6')\end{aligned}\right\}(c')$$

Whenever the equation $B = 0$ or $B' = 0$ is impossible, system (c) will be impossible, and system (a) will be equivalent to system (b).

Ex. 1. Solve the system
$$\left.\begin{aligned} x^3 - y^3 &= 27, \quad &(1)\\ x - y &= 3, \quad &(2)\end{aligned}\right\}(a)$$

Dividing (1) by (2) we obtain the *integral* equation (3); hence, as $B' = 0$, or $3 = 0$, is impossible, system (a) is equivalent to (b).

$$\left.\begin{aligned} x^2 + xy + y^2 &= 9, \quad &(3)\\ x - y &= 3. \quad &(4)\end{aligned}\right\}(b)$$

Ex. 2. Solve the system
$$\left.\begin{aligned} x^4 + x^2y^2 + y^4 &= 7371, \quad &(1)\\ x^2 - xy + y^2 &= 63. \quad &(2)\end{aligned}\right\}(a)$$

Divide (1) by (2), $\quad x^2 + xy + y^2 = 117. \quad (3)$

Add (2) and (3), $\quad x^2 + y^2 = 90, \quad (4)$

Subtract (2) from (3), $\quad 2\,xy = 54. \quad (5)$

(4) and (5) form system (b).

Since $63 = 0$ is impossible, by *division* (§ 309) and *addition* (§ 204), system (b) is equivalent to (a).

Ex. 3. Solve the system
$$\left.\begin{aligned} x^2y + xy^2 &= 30, \quad &(1)\\ x + y &= 5. \quad &(2)\end{aligned}\right\}(a)$$

Divide (1) by (2), $\quad xy = 6. \quad (3)$

Equations (2) and (3) form a system equivalent to (a).

Exercise 117.

Solve each of the following systems:

1. $\left.\begin{aligned} x^3 + y^3 &= 3473,\\ x + y &= 23.\end{aligned}\right\}$

2. $\left.\begin{aligned} x^3 - y^3 &= 218,\\ x - y &= 2.\end{aligned}\right\}$

3. $\left.\begin{aligned} x^3 - y^3 &= 988,\\ x - y &= 4.\end{aligned}\right\}$

4. $\left.\begin{aligned} x^3 - y^3 &= 2197,\\ x - y &= 13.\end{aligned}\right\}$

5. $\left.\begin{aligned} x^4 + x^2y^2 + y^4 &= 2128,\\ x^2 + xy + y^2 &= 76.\end{aligned}\right\}$

6. $\left.\begin{aligned} x^4 + x^2y^2 + y^4 &= 243,\\ x^2 - xy + y^2 &= 9.\end{aligned}\right\}$

7. $\left.\begin{aligned} x^4 + x^2y^2 + y^4 &= 91,\\ x^2 + xy + y^2 &= 13.\end{aligned}\right\}$

8. $\left.\begin{aligned} x^4 + x^2y^2 + y^4 &= 2923,\\ x^2 - xy + y^2 &= 37.\end{aligned}\right\}$

9. $\left.\begin{aligned} x^4 + x^2y^2 + y^4 &= 7371,\\ x^2 - xy + y^2 &= 63.\end{aligned}\right\}$

10. $\left.\begin{aligned} x^3 - y^3 &= 56,\\ x^2 + xy + y^2 &= 28.\end{aligned}\right\}$

11. $\left.\begin{array}{l} x^3 + y^3 = 126, \\ x^2 - xy + y^2 = 21. \end{array}\right\}$

12. $\left.\begin{array}{l} x + y - \sqrt{xy} = 7, \\ x^2 + y^2 + xy = 133. \end{array}\right\}$

In the next four systems apply § 304 first.

13. $\left.\begin{array}{l} x + y = 5, \\ 4\,xy = 12 - x^2y^2. \end{array}\right\}$

14. $\left.\begin{array}{l} x^2y + xy^2 = 180, \\ x^2y^2 = 400. \end{array}\right\}$

15. $\left.\begin{array}{l} x + y = 1, \\ x^2y^2 + 13\,xy + 12 = 0. \end{array}\right\}$

16. $\left.\begin{array}{l} 5\,x^2 - 5\,y^2 = x + y, \\ 3\,x^2 - 3\,y^2 = x - y. \end{array}\right\}$

In the next four systems let the unknowns be the reciprocals of x and y, and let $v = 1/x$ and $w = 1/y$.

17. $\left.\begin{array}{l} 2/x + 1/y = 1, \\ \dfrac{2}{x^2} + \dfrac{3}{xy} + \dfrac{1}{y^2} = 5. \end{array}\right\}$

18. $\left.\begin{array}{l} 1/x + 1/y = 2, \\ 1/x^2 + 1/y^2 = 20. \end{array}\right\}$

19. $\left.\begin{array}{l} 3/x^2 - 1/y^2 = 1, \\ \dfrac{5}{x^2} - \dfrac{1}{xy} + \dfrac{2}{y^2} = 3. \end{array}\right\}$

20. $\left.\begin{array}{l} 1/x^2 - 1/(4\,y^2) = 3, \\ \dfrac{1}{x^2} - \dfrac{1}{xy} + \dfrac{1}{4\,y^2} = 9. \end{array}\right\}$

21. $\left.\begin{array}{l} \dfrac{x+y}{x-y} + \dfrac{x-y}{x+y} = \dfrac{5}{2}, \\ x^2 + y^2 = 20. \end{array}\right\}$

22. $\left.\begin{array}{l} x^2/y + y^2/x = 9/2, \\ x + y = 3. \end{array}\right\}$

310. It should be observed that the methods given in this chapter are applicable only to special systems of quadratic and higher equations, and do not enable us to solve a system of *any* two quadratic equations; for the equation derived by eliminating one unknown will, in general, be above the second degree in the other unknown, and we have not yet learned how to solve an equation of a higher degree than the second, except in very special cases.

E.g., consider the system

$$x^2 + x + y = 3, \;\; x^2 + y^2 = 5. \tag{a}$$

Solving the first equation for y and substituting its value in the second, we have

$$x^2 + (3 - y - x)^2 = 5,$$

or

$$x^4 + 2\,x^3 - 4\,x^2 - 6\,x + 4 = 0. \tag{1}$$

Equation (1), which is of the fourth degree, cannot be solved by any methods which have been given in the previous chapters.

Exercise 118.

1. The difference of two numbers is 7, and the sum of their squares is 169. Find the numbers.

2. The sum of the squares of two numbers is 130, and the difference of their squares is 32. Find the numbers.

3. The sum of two numbers is 39, and the sum of their cubes is 17,199. Find the numbers.

4. A person bought some fine sheep for $ 360, and found that if he had bought 6 more for the same money, he would have paid $ 5 less for each. How many did he buy, and what was the price of each?

5. If the length and breadth of a rectangle were each increased by 1 yard, the area would be 48 square yards; if they were each diminished by 1 yard, the area would be 24 square yards. Find the length and breadth.

6. The numerator and denominator of one fraction are each greater by 1 than those of another, and the sum of the two is $1\frac{5}{12}$; if the numerators were interchanged, the sum of the fractions would be $1\frac{1}{2}$. Find the fractions.

7. For a journey of 108 miles, 6 hours less would have sufficed, had the traveller gone 3 miles an hour faster. At what rate did he travel?

8. The hypotenuse of a right-angled triangle is 20 feet, and its area is 96 square feet. Find the length of the other two sides.

9. A number is divided into two parts such that the sum of the first and the square of the second is twice the sum of the second and the square of the first; and the sum of the number and the first part is 4 more than twice the second. Find the number.

10. The small wheel of a bicycle makes 135 revolutions more than the large wheel in a distance of 260 yards; if

the circumference of each were one foot more, the small wheel would make 27 revolutions more than the large wheel in a distance of 70 yards. Find the circumference of each wheel.

11. A man bought 6 ducks and 2 turkeys for \$ 15. For \$ 14 he could buy 4 more ducks than he could turkeys for \$ 9. Find the price of each.

12. The sum of the cubes of two numbers is 407, and the sum of their squares exceeds their product by 37. Find the numbers.

13. A rectangular field contains 160 square rods. If its length be increased by 4 rods, and its breadth by 3 rods, its area will be increased by 100 square rods. Find the length and breadth of the field.

14. A man rows down stream 12 miles in 4 hours' less time than it takes him to return. Should he row at twice his ordinary rate, his rate down stream would be 10 miles an hour. Find his rate in still water, and the rate of the stream.

15. The sum of two numbers is 7, and the sum of their fourth powers is 641. Find the numbers.

16. A gentleman left \$ 210 to 3 servants to be divided in continued proportion, so that the first should have \$ 90 more than the last. Find the legacy of each.

17. From a sheet of paper 14 inches long, a border of uniform width is cut away all round it, and the area is thereby reduced $\frac{5}{8}$; but had the sheet been 3 inches narrower, and a border of the same width had been cut away, the area would have been reduced $\frac{5}{7}$. Find the breadth of the paper, and the width of the border cut away.

18. A and B set out from the same place, and travel in the same direction at uniform rates. B starts 5 hours after

A, and overtakes him after travelling 100 miles. Had their rates of travelling been a mile per hour less, B would have overtaken A after travelling 60 miles. Find their rates.

19. A man has to travel a certain distance, and, when he has travelled 40 miles, he increases his speed 2 miles per hour. If he had travelled with his increased speed during the whole journey, he would have arrived 40 minutes earlier; but if he had continued at his original speed, he would have arrived 20 minutes later. Find the whole distance he had to travel, and his original speed.

20. A cubical tank contains 512 cubic feet of water. It is required to enlarge the tank, the depth remaining the same, so that it shall contain 7 times as much water as before, subject to the condition that the length added to one side of the base shall be 4 times that added to the other. Find the sides of the new rectangular base.

CHAPTER XXIV

INEQUALITIES

311. An **inequality** is the statement that one number is greater or less than another, as $6 > 4$, $-3 < -2$. See §§ 7 and 31.

312. When a and b are real, in § 31 we agreed to say that:

$a > b$, when $a - b$ is *positive;*

and $a < b$, when $a - b$ is *negative.*

The statement '*$a - b$ is positive*' is expressed in symbols by $a - b > 0$; and '*$a - b$ is negative*' by $a - b < 0$.

In this chapter we shall not consider imaginary or complex numbers.

313. Two inequalities are said to be **like** or **unlike** *in species* according as they *do* or *do not* have the *same* sign of inequality.

E.g., the inequalities $8 > 4$ and $a > b$ are *like* in species; while $2 < 3$ and $a > b$ are *unlike* in species.

If $a > b$; then, *conversely*, $b < a$.

The inequality $a > b$ and its converse $b < a$ are *unlike.*

314. Principles of inequalities.

(i) *If one number* $>$ *a second, and this second number* $>$ *a third, then the first number* $>$ *the third number.*

That is, if $a > b$ and $b > c$, then $a > c$.

(ii) *If the same number is added to both members or subtracted from both members of an inequality, the derived inequality will be* **like** *the given one.*

That is. if $a > b$, then $a \pm m > b \pm m$.

(iii) *If the corresponding members of two or more* **like** *inequalities are added, the derived inequality will be* **like** *the given ones.*

That is, if $a > b$ and $c > d$, then $a + c > b + d$.

(iv) *If both members of an inequality are multiplied, or divided, by the same positive number, the derived inequality will be* **like** *the given one.*

That is, if $a > b$, then $a(^+n) > b(^+n)$, or $a \div {}^+n > b \div {}^+n$.

(v) *If both members of an inequality are multiplied, or divided, by the same negative number, the derived inequality will be* **unlike** *the given one.*

That is, if $a > b$, then $a(^-n) < b(^-n)$, or $a \div {}^-n < b \div {}^-n$.

(vi) *If all the members of two or more* **like** *inequalities are positive, and if the corresponding members are multiplied together, the derived inequality will be* **like** *the given ones.*

That is, if $^+a_1 > {}^+b_1$, $^+a_2 > {}^+b_2$, $\cdots$,

then $$^+a_1 \cdot {}^+a_2 \cdots > {}^+b_1 \cdot {}^+b_2 \cdots.$$

(vii) *If both members of an inequality are positive, and they are raised to the same positive integral power, the derived inequality will be* **like** *the given one.*

That is, if $^+a > {}^+b$, then $(^+a)^n > (^+b)^n$, where n is a positive integer.

(viii) *If the same principal roots of both members of an inequality are taken, the derived inequality will be* **like** *the given one.*

That is, if $a > b$, then $\sqrt[r]{a} > \sqrt[r]{b}$.

Proof of (i). $(a - b) + (b - c) \equiv a - c$.

Hence, if $a - b > 0$ and $b - c > 0$, then $a - c > 0$;

that is, if $$a > b \text{ and } b > c, \text{ then } a > c.$$

Proof of (ii). $a - b \equiv (a \pm m) - (b \pm m)$.

Hence, if $a > b$, then $a \pm m > b \pm m$.

The proof of the other principles is left as an exercise for the pupil.

315. The following principle is often useful in proving inequalities:

*If **a** and **b** are unequal and real,* $a^2+b^2>2ab$.

Proof. $$(a-b)^2>0,$$

or $$a^2-2ab+b^2>0. \tag{1}$$

Adding $2ab$ to each member, by (ii) of § 314 we obtain

$$a^2+b^2>2ab, \text{ when } a\neq b. \tag{2}$$

Observe that a^2 and b^2 are both positive.

Ex. 1. Prove $(x+y)/2>\sqrt{xy}$, if $x>0$, $y>0$, and $x\neq y$.

If in (2) we put x for a^2 and y for b^2, we obtain

$$x+y>2\sqrt{xy}.$$

Hence, by (iv), $(x+y)/2>\sqrt{xy}$, where $x>0$, $y>0$, and $x\neq y$.

Ex. 2. $a^3+b^3>a^2b+ab^2$, if $a+b>0$ and $a\neq b$.

From (1), $\quad a^2-ab+b^2>ab.$ by (ii)

Multiply by $a+b$, $\quad a^3+b^3>a^2b+ab^2.$ by (iv)

Ex. 3. The sum of any *positive* number, except 1, and its reciprocal is greater than 2.

Let the number be n; then in (2), putting n for a^2 and $1/n$ for b^2, we obtain

$$n+1/n>2.$$

316. The following examples illustrate some of the uses of the principles of inequalities:

Ex. 1. For what values of x is $(5x-7)/3>(2-3x)/5$? (1)

Multiply by 15, $\quad 25x-35>6-9x.$ by (iv)

Transpose, $\quad 34x>41.$ by (ii)

Divide by 34, $\quad x>41/34.$ by (iv)

Hence (1) is satisfied for any value of x greater than $41/34$.

Ex. 2. For what values of x is $x^2-4x+3>-1$? (1)

Add 1, $\quad x^2-4x+4>0$, or $(x-2)^2>0.$ by (ii)

Hence (1) is satisfied when $(x-2)^2>0$, *i.e.*, when x has any real value except 2.

Ex. 3. Find what values of x satisfy the inequalities

$$\left.\begin{array}{lr} 4x - 6 < 2x + 4, & (1) \\ \text{and} \quad 2x + 4 > 16 - 2x. & (2) \end{array}\right\}$$

From (1), $2x < 10$, or $x < 5$. by (ii), (iv)

From (2), $4x > 12$, or $x > 3$. by (ii), (iv)

Hence (1) and (2) are satisfied by any value of x between 3 and 5.

Ex. 4. Find what values of x satisfy the inequality

$$x^2 - 7x < 8. \qquad (1)$$

Subtract 8, $x^2 - 7x - 8 < 0$, or $(x - 8)(x + 1) < 0$. by (ii)

The product $(x - 8)(x + 1)$ will be *negative*, when, and only when, one factor is positive and the other negative.

One of these factors will be positive and the other negative when x has any value between -1 and 8, and only then.

Hence (1) is satisfied by any value of x between -1 and 8.

Ex. 5. Find what values of x and y satisfy the inequality

$$\left.\begin{array}{lr} 3x + 2y > 5, & (1) \\ \text{and the equation} \quad 5x + 7y = 12. & (2) \end{array}\right\}$$

Multiply (1) by 5, $15x + 10y > 25$. (3)

Multiply (2) by 3, $15x + 21y = 36$. (4)

Subtract (4) from (3), $-11y > -11$, or $y < 1$. by (v)

Multiply (1) by 7, $1x + 14y > 35$. (5)

Multiply (2) by 2, $10x + 14y = 24$. (6)

Subtract (6) from (5), $11x > 11$, or $x > 1$.

Hence any solution of equation (2) in which $x > 1$ and $y < 1$ will satisfy both (1) and (2).

Exercise 119.

If the letters denote unequal positive numbers, prove:

1. $a^2 + b^2 + c^2 > ab + ac + bc$. (1)

Use the relation $a^2 + b^2 > 2ab$.

2. $a^3 + b^3 > a^2b + ab^2$. (2)

3. $\dfrac{a+b}{2} > \dfrac{2ab}{a+b}$; $\dfrac{a}{b^2} + \dfrac{b}{a^2} > \dfrac{1}{b} + \dfrac{1}{a}$. (3)

4. $a^3 + b^3 + c^3 > (a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2)/2$.

5. $am + bn + cr < 1$, if $a^2 + b^2 + c^2 = 1$, and $m^2 + n^2 + r^2 = 1$.

Find the limits between which the values of x must lie to satisfy each of the following inequalities:

6. $6x > \frac{3}{2}x + 18$.

7. $\frac{7}{5}x - \frac{5}{3}x > \frac{1}{3}x - 3$.

8. $-2(x+7) > -16$.

9. $x^2 - 5x > -4$.

10. $x^2 + x > 12$.

11. $(x+2)/(x-3) > 0$.

12. $(x-7)/(x+4) < 0$.

13. $3(x+7)/5 > 5(x-3)/7$.

14. If $5x - 6 < 3x + 8$ and $2x + 1 < 3x - 3$, show that the values of x lie between 4 and 7.

15. If $3x - 2 > \frac{1}{2}x - \frac{4}{5}$ and $\frac{7}{6} - \frac{5}{4}x < 8 - 2x$, show that the values of x lie between 12/25 and 82/9.

Find what values of x and y will satisfy each of the following systems:

16. $\left.\begin{array}{r} 2x+3y=4, \\ x-y>2. \end{array}\right\}$ 17. $\left.\begin{array}{r} 3x-y=6, \\ 2x+y>4. \end{array}\right\}$ 18. $\left.\begin{array}{r} 4x-2y=6, \\ 2x-3y>5. \end{array}\right\}$

19. Show that (1) in example 1 holds, if a, b, and c are real and either $a \neq b$, or $a \neq c$, or $b \neq c$.

20. Show that (2) and (3) in examples 2 and 3 hold, if a and b are real and unequal and $a + b > 0$.

CHAPTER XXV

RATIO AND PROPORTION

317. The **ratio** of one number to another is the quotient of the first divided by the second.

The dividend is called the **first term**, or the **antecedent**, of the ratio; and the divisor, the **second term**, or **consequent**.

The ratio of a to b is written $\frac{a}{b}$, a/b, $a \div b$, or $a : b$, each of which forms can be read 'a is to b' or 'a by b.'

The ratio of 8 to 2 is $8/2$, or 4; the ratio of 7 to 5 is $7/5$.

It is clear that a ratio is arithmetically greater than, equal to, or less than 1, according as its first term is arithmetically greater than, equal to, or less than, the second.

318. Since *a ratio is a fraction*, all the properties of fractions belong to ratios in whatever form the ratios are written.

Thus $\qquad a : b \equiv am : bm$, or $a/b \equiv am/(bm)$; § 172

and $\qquad a : b \equiv (a \div m) : (b \div m)$, or $a/b \equiv (a \div m)/(b \div m)$. § 173

Two ratios can be compared by reducing them as fractions to a common denominator.

Ex. 1. Which is the greater, $3 : 11$ or $5 : 19$?

$$3 : 11 = 3/11 = 57/209, \text{ and } 5 : 19 = 5/19 = 55/209;$$

hence the ratio $3 : 11 >$ the ratio $5 : 9$.

Ex. 2. $(a : b)^3 \equiv a^3 : b^3$; $\sqrt{a : b} \equiv \sqrt{a} : \sqrt{b}$. §§ 186, 225.

319. By § 91, $(a : b)(c : d)(e : f) \equiv ace : bdf$.

The ratio $ace : bdf$ is said to be **compounded** of the ratios $a : b$, $c : d$, and $e : f$.

320. The **inverse** of a ratio is its reciprocal.

Hence the inverse of the ratio $a:b$ is the ratio $b:a$ (§ 182).

321. By § 183, $a/b : c/d \equiv ad : bc$.

Hence the ratio of any two fractions can be expressed by the ratio of two integers.

322. Two numbers are said to be *commensurable* or *incommensurable* **with each other** according as their ratio *can* or *cannot* be expressed by the ratio of two integers.

E.g., $\sqrt{2}$ and 5 are *incommensurable with each other*, so also are $\sqrt{3}$ and $\sqrt[3]{5}$. The incommensurable numbers $3\sqrt{2}$ and $7\sqrt{2}$ are commensurable *with each other;* for their ratio is $3/7$. Compare § 224.

323. Ratio of concrete quantities. If A and B are two concrete quantities of the same kind, whose numerical measures in terms of the same unit are the numbers a and b, then the ratio of A to B is defined to be the ratio of a to b.

Exercise 120.

Find the simplest expressions for the following ratios:

1. $6a$ to $12a^2$.
2. $3a^2x/5$ to $6ax^2/7$.
3. $1/a$ to $1/b$.
4. a/x to c/y.
5. $a/(x-2)$ to $3/(x-2)^2$.
6. $9/(a-b)^2$ to $6/(a-b)$.
7. Write as a ratio $(2x:3y)^2$; $(2a:b)^3$; $(a:c)^5$; $\sqrt[3]{a:b}$.

Find the ratio compounded of:

8. The ratio $25:8$ and the square of the ratio $4:3$.
9. The ratio $32:27$ and the cube of the ratio $3:2$.
10. The ratio $6:7$ and the square root of the ratio $25:36$.
11. Arrange the ratios $5:6$, $7:8$, $41:48$, and $31:36$ in descending order of magnitude.

12. For what value of x will the ratio $15+x:17+x$ be equal to $1/2$?

13. What number must be added to each of the terms of the ratio $3:4$ to make it equal to the ratio $25:32$?

Let $x =$ the number to be added; then

$$(3+x)/(4+x) = 25/32.$$

14. Find two numbers in the ratio of 5 to 6, whose sum is 121.

15. Which is the greater ratio, $5:7$ or $5+2:7+2$?

16. Which is the greater ratio, $7:5$ or $7+2:5+2$?

PROPORTION.

324. Four quantities are said to be **in proportion** when the ratio of the first to the second is equal to the ratio of the third to the fourth.

An equality whose members are two equal ratios is called a **proportion**. Thus, if

$$a:b=c:d, \tag{1}$$

then a, b, c, and d are *in proportion*, or are *proportional*, and equation (1) is a *proportion*.

A proportion can be written in the form

$$a/b=c/d,\ a:b=c:d, \text{ or } a:b::c:d,$$

each of which is read 'a by b is equal to c by d,' or 'a is to b as c is to d.'

The four numbers in a proportion are called the **proportionals**, or the **terms**, of the proportion.

The first and fourth terms are called the **extremes**, and the second and third the **means**.

E.g., a and d are the extremes, and b and c are the means in the proportion

$$a:b=c:d.$$

In (1), d is called the *fourth proportional* to a, b, and c.

325. The following theorem and its converse in the next article are the two fundamental principles in proportion.

In any proportion the product of the extremes is equal to the product of the means.

That is, if $$a : b = c : d, \qquad (1)$$

then $$ad = bc. \qquad (2)$$

Proof. Clearing (1) of fractions, we obtain (2).

Ex. The first, second, and fourth terms of a proportion are c^2, $2a$, and $5b$ respectively; find the third term.

Let $x =$ the third term of the proportion;

then $$c^2 : 2a = x : 5b.$$

$$\therefore 2ax = 5bc^2, \text{ or } x = 5bc^2/(2a).$$

326. Conversely, *if the product of one set of two numbers is equal to the product of another set of two numbers, either set can be made the extremes and the other set the means of a proportion.*

Proof. Let $$ad = bc. \qquad (1)$$

Divide (1) by db, $\quad a : b = c : d$, or $c : d = a : b$.

Divide (1) by dc, $\quad a : c = b : d$, or $b : d = a : c$.

Divide (1) by ab, $\quad d : b = c : a$, or $c : a = d : b$.

Divide (1) by ac, $\quad d : c = b : a$, or $b : a = d : c$.

From this principle it follows that —

(i) *A proportion is proved when it is proved that the product of its extremes is equal to the product of its means.*

(ii) *In a given proportion, we can interchange the means, or the extremes, or we can take the means as extremes and the extremes as means.*

327. If $$a : b = c : d,$$

then $$ma : mb = nc : nd, \qquad \S\ 172$$

and $$ma : nb = mc : nd. \qquad \S\S\ 6, 91$$

328. *Any* **proportion, as** $\quad a : b = c : d, \quad$ (1)
can be taken by

(i) **inversion;** *that is,* $\quad b : a = d : c, \quad$ (2)

(ii) **alternation;** *that is,* $\quad a : c = b : d, \quad$ (3)

(iii) **addition;** *that is,* $\quad a + b : a = c + d : c, \quad$ (4)

or $\quad a + b : b = c + d : d, \quad$ (5)

(iv) **subtraction;** *that is,* $\quad a - b : a = c - d : c, \quad$ (6)

or $\quad a - b : b = c - d : d, \quad$ (7)

(v) **addition and subtraction;** *that is,*

$$a + b : a - b = c + d : c - d. \tag{8}$$

Proof. From (1), $\quad ad = bc. \quad$ (1′)

Add bd to (1′), $\quad (a + b)d = (c + d)b. \quad$ (2′)

Add $-bd$ to (1′), $\quad (a - b)d = (c - d)b. \quad$ (3′)

By § 326, from (1′), we have (2) and (3); from (2′), (5); and from (3′), (7).

Dividing (2′) by (1′), we obtain (4).
Dividing (3′) by (1′), we obtain (6).
Dividing (2′) by (3′), we obtain (8).

Observe that (2) and (3) can be obtained from (1) by (ii) of § 326.

329. *The products or the quotients of the corresponding terms of two proportions are proportional.*

That is, if $\quad a : b = c : d, \quad$ (1)

and $\quad a' : b' = c' : d', \quad$ (2)

then $\quad aa' : bb' = cc' : dd', \quad$ (3)

and $\quad a/a' : b/b' = c/c' : d/d'. \quad$ (4)

Proof. Multiplying (1) by (2), by §§ 6 and 91 we obtain (3).

Dividing (1) by (2), since $\dfrac{a/b}{a'/b'} \equiv \dfrac{a/a'}{b/b'}$, we obtain (4).

330. *Like powers or like principal roots of proportionals are proportional.*

That is, if $$\boldsymbol{a : b = c : d}, \tag{1}$$
then $$\boldsymbol{a^n : b^n = c^n : d^n}, \tag{2}$$
and $$\boldsymbol{\sqrt[r]{a} : \sqrt[r]{b} = \sqrt[r]{c} : \sqrt[r]{d}}. \tag{3}$$

Proof. By §§ 128 and 186, from (1) we obtain (2).
By §§ 221 and 225, from (1) we obtain (3).

331. *In a series of equal ratios the sum of the antecedents is to the sum of the consequents as any one antecedent is to its consequent.*

That is, if $$\boldsymbol{a : b = c : d = e : f = \cdots}, \tag{1}$$
then $\boldsymbol{a + c + e + \cdots : b + d + f + \cdots = a : b = c : d = \cdots}$.

Proof. Let $a/b = r$; then $c/d = r$, $e/f = r$, $\cdots$;

hence $$a = br,\ c = dr,\ e = fr,\ \cdots.$$

Adding the members of these equations, by § 6, we obtain

$$a + c + e + \cdots = (b + d + f + \cdots)\, r.$$

$$\therefore\ \frac{a + c + e + \cdots}{b + d + f + \cdots} = r = \frac{a}{b} = \frac{c}{d} = \cdots.$$

332. A *general* and *easy* method for proving a proportion is to represent the value of one of the equal ratios in the given proportion by a single letter, as was done in the last section.

Ex. 1. Given $a : b = c : d$, prove that

$$a^2 + ab : c^2 + cd = b^2 - 2\,ab : d^2 - 2\,cd. \tag{1}$$

Let $$a/b = r;\ \text{then } c/d = r;$$

then $$a = br, \text{ and } c = dr.$$

Substituting these values of a and c in each ratio of (1), we have

$$\frac{a^2 + ab}{c^2 + cd} = \frac{b^2r^2 + b^2r}{d^2r^2 + d^2r} = \frac{b^2(r^2 + r)}{d^2(r^2 + r)} = \frac{b^2}{d^2},$$

and $$\frac{b^2 - 2\,ab}{d^2 - 2\,cd} = \frac{b^2 - 2\,b^2r}{d^2 - 2\,d^2r} = \frac{b^2(1 - 2\,r)}{d^2(1 - 2\,r)} = \frac{b^2}{d^2}.$$

Hence the ratios in (1) are equal.

Ex. 2. Given $a:b=c:d=e:f$, prove that

$$a^3+c^3+e^3:b^3+d^3+f^3=ace:bdf. \qquad (1)$$

Let $\quad a/b=r$; then $c/d=r$, and $e/f=r$.

Hence $\quad a=br$, $c=dr$, and $e=fr$.

Substituting the values of a, c, and e first in the product of the extremes, and then in the product of the means, we obtain

$$(a^3+c^3+e^3)bdf=(b^3+d^3+f^3)r^3bdf,$$

and

$$(b^3+d^3+f^3)ace=(b^3+d^3+f^3)r^3bdf.$$

That is, the product of the extremes in (1) is equal to the product of the means; hence, by (i) of § 326, (1) is proved.

333. A **continued proportion** is a proportion in which the consequent of each ratio is the antecedent of the following ratio. Thus a, b, c, $d \cdots$ are in continued proportion if

$$a:b=b:c=c:d=\cdots.$$

If $a:b=b:c$, then b is called a **mean proportional** *between* a and c, and c is called a **third proportional** to a and b.

If $a:b=b:c=c:d$, then b and c are called the **two mean proportionals** between a and d.

334. *The mean proportional between two numbers is equal to the square root of their product.*

Proof. If $a:b=b:c$, then $b^2=ac$, or $b=\sqrt{ac}$.

Exercise 121.

From each of the following products form four different proportions and their converses:

1. $xy=mn$. 2. $6\times 3=2\times 9$. 3. $a^2-b^2=x^2-y^2$.

Find the fourth proportional to the three numbers:

4. a, ab, c. 5. a^2, $2ab$, $3b^2$. 6. x^3, xy, $5x^2y$.

Find the third proportional to the two numbers:

7. a^2b, ab. 8. x^3, $2x^2$. 9. $3x$, $6xy$. 10. 1, x.

Find a mean proportional between the two numbers:

11. a^2, b^2.

12. $2x^3, 8x$.

13. $12ax^2, 3a^3$.

14. $27a^2b^3, 3b$.

If $a:b=c:d$, show that

15. $ac:bd=c^2:d^2$.

16. $a^2:c^2=a^2-b^2:c^2-d^2$.

17. $2a+3c:3a+2c=2b+3d:3b+2d$.

18. $la+mb:pa+gb=lc+md:pc+gd$.

19. $a:a+c=a+b:a+b+c+d$.

20. $a^2+ab+b^2:a^2-ab+b^2=c^2+cd+d^2:c^2-cd+d^2$.

21. $a+b:c+d=\sqrt{a^2+b^2}:\sqrt{c^2+d^2}$.

22. $\sqrt{a^2+b^2}:\sqrt{c^2+d^2}=\sqrt[3]{a^3+b^3}:\sqrt[3]{c^3+d^3}$.

23. $a^2c+ac^2:b^2d+bd^2=(a+c)^3:(b+d)^3$.

24. $\sqrt[n]{a^n+b^n}:\sqrt[n]{c^n+d^n}=\sqrt[r]{a^r-b^r}:\sqrt[r]{c^r-d^r}$.

25. If $a:b=b:c$, prove that $a:c=a^2:b^2$.

26. If $a:b=b:c=c:d$, prove that $a:d=a^3:b^3$.

Let $r=a\div b$; then $a=br,\ b=cr,\ c=dr$.

$\therefore abc=bcdr^3$. $\quad \therefore a\div d=r^3=a^3\div b^3$.

27. If a, b, c, d be any four numbers, find what number must be added to each to make the results proportional.

28. Two numbers are in the ratio of 3 to 8, and the sum of their squares is 3577; find them.

29. The ages of two persons are as $3:4$, and 30 years ago they were as $1:3$; find their present ages.

30. The sides of a triangle are as $3:4:5$, and the perimeter is 480 yards; find the sides.

31. Divide the number 14 into two such parts that the quotient of the greater divided by the less shall be to the quotient of the less divided by the greater as 16 to 9.

32. Show that the ratio of any two fractions, not involving surds, can be expressed by the ratio of two whole numbers.

33. Express the ratio of $5\frac{1}{2}$ to $7\frac{4}{13}$ by the ratio of two whole numbers.

34. Express the ratio of $17\frac{1}{4}$ to $14\frac{2}{3}$ by the ratio of two whole numbers.

35. The sum of two numbers is 8, and their product is to the sum of their squares as 3 to 10. What are the numbers?

36. The sum of two numbers is 10, and the sum of their squares is to the square of their sum as 13 to 25. What are the numbers?

37. A hare is pursued by a greyhound, and is 60 of her own leaps before him. The hare takes 3 leaps in the time that the greyhound takes 2; but the greyhound goes as far in 3 leaps as the hare does in 7. In how many leaps will the greyhound catch the hare?

Let $x =$ the number of leaps taken by the greyhound,

and $y =$ the number of leaps taken by the hare in the same time;

then $$x : y = 2 : 3,$$

and $$x + 60 : y = 7 : 3.$$

CHAPTER XXVI

THEORY OF EXPONENTS

335. Hitherto we have defined and used only positive integers as exponents. It is, however, found convenient to extend the meaning of an exponent so that we can use zero, a fraction, or a negative number, as an exponent.

As it is desirable that all exponents should obey the same laws, we shall fix the meaning of (*i.e.*, define) any new exponent by imposing the restriction that all exponents must obey the fundamental law,

$$a^m \times a^n \equiv a^{m+n}. \tag{1}$$

E.g., to find the meaning of $a^{\frac{1}{3}}$, we have by law (1),

$$a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \equiv a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} \equiv a^{\frac{3}{3}} \equiv a;$$

that is, $\quad (a^{\frac{1}{3}})^3 \equiv a; \quad \therefore\ a^{\frac{1}{3}} \equiv$ the cube root of a. § 213

Again, to find the meaning of $a^{\frac{3}{2}}$, we have by law (1),

$$a^{\frac{3}{2}} \cdot a^{\frac{3}{2}} \equiv a^{\frac{3}{2}+\frac{3}{2}} \equiv a^{\frac{6}{2}} \equiv a^3;$$

that is, $\quad (a^{\frac{3}{2}})^2 \equiv a^3; \quad \therefore\ a^{\frac{3}{2}} \equiv$ the square root of a^3. § 213

336. Meaning of a positive fractional exponent.

Let r and s denote any positive integers; then by the fundamental law of exponents, we have

$$a^{\frac{r}{s}} \cdot a^{\frac{r}{s}} \cdots \text{to } s \text{ factors} \equiv a^{\frac{r}{s}+\frac{r}{s}+\cdots \text{to } s \text{ terms}}$$
$$\equiv a^{\frac{rs}{s}} \equiv a^r;$$

that is, $\quad (a^{\frac{r}{s}})^s \equiv a^r; \quad \therefore\ a^{\frac{r}{s}} \equiv$ the sth root of a^r; § 213

that is, $a^{\frac{r}{s}}$ is only another way of writing the sth root of the rth power of a.

Hence a **positive fractional exponent** denotes a root of a power of its base. The denominator indicates the root, and the numerator the power.

Thus $a^{\frac{1}{s}}$ denotes the sth root of a.

337. Using $a^{\frac{r}{s}}$ to denote only the *principal* sth *root* of a^r, we have

$$\boldsymbol{a^{\frac{r}{s}} \equiv \sqrt[s]{a^r} \equiv (\sqrt[s]{a})^r}, \qquad \text{§§ 220, 226}$$

and $$\boldsymbol{a^{\frac{1}{s}} \equiv \sqrt[s]{a}}.$$

Ex. 1. $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$; $\quad 4^{\frac{3}{2}} = (\sqrt{4})^3 = 2^3 = 8$.

Ex. 2. $8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = 2^2 = 4$.

Ex. 3. $(-32)^{\frac{4}{5}} = (\sqrt[5]{-32})^4 = (-2)^4 = 16$.

338. Meaning of zero as exponent.

By the fundamental law of exponents, we have

$$a^m \cdot a^0 \equiv a^{m+0} \equiv a^m;$$

$$\therefore\ a^0 = a^m/a^m = 1.$$

That is, *any base, except zero, with zero as an exponent is equal to positive one.*

Observe that a^0 is only another way of writing a^m/a^m, or 1.

E.g., $$a^0 \equiv a/a \equiv a^2/a^2 \equiv a^3/a^3 \equiv \cdots \equiv 1.$$

339. Meaning of a negative exponent.

Let n denote any positive integer or fraction; then by the fundamental law of exponents, we have

$$a^n \cdot a^{-n} \equiv a^{n+(-n)} \equiv a^0 \equiv 1.$$

$$\therefore\ \boldsymbol{a^{-n} \equiv 1/a^n}.$$

That is, a^{-n} is only another way of writing the reciprocal of a^n, or of denoting that a^n is to be used as a divisor.

E.g., $3^{-2} = 1/3^2 = 1/9$; $(-2)^{-3} = 1/(-2)^3 = -1/8$.

NOTE. The *arithmetic value* of an exponent denotes a power, or a root of a power, of its base; and its *quality* denotes whether this power or root is to be used as a *factor* or *divisor*.

Fractional and negative exponents express no new ideas, and are not necessary to the notation of algebra; but they are very convenient, and greatly facilitate many operations. *Fractional* exponents simply afford another way of writing a root of a power; and *negative* exponents, another way of writing a *divisor*.

340. Hereafter in this chapter we shall use $a^{\frac{r}{s}}$ to denote only the *principal* sth root of a^r, or, what is the same thing, the rth power of the *principal* sth root of a.

Observe that r can be either positive or negative, but that s is always positive.

Ex. 1. $8^{-\frac{2}{3}} = 2^{-2} = 1/2^2 = 1/4.$

Ex. 2. $(-27)^{-\frac{4}{3}} = (-3)^{-4} = 1/(-3)^4 = 1/81.$

Ex. 3. $(-32)^{-\frac{6}{5}} = (-2)^{-6} = 1/64.$

Note the advantage of first extracting the root in these examples.

341. A base with any exponent is called an **exponential expression**; as, 3^2, a^x, $(x+y)^a$, c^{x+y}.

342. The quality of an exponent can be changed if the sign before the exponential expression is changed from $\times$ to $\div$, or from $\div$ to $\times$.

Proof. $$a \times b^{-n} \equiv a \times (1/b^n) \equiv a \div b^n. \tag{1}$$

Also $$a \div b^{-n} \equiv a \div (1/b^n) \equiv a \times b^n. \tag{2}$$

343. *Any exponential factor can be transferred from the dividend to the divisor, or from the divisor to the dividend, if the quality of its exponent is changed from* $+$ *to* $-$, *or from* $-$ *to* $+$.

Proof. This is the converse of (1) and (2) in § 342.

Or this operation is simply multiplying both dividend and divisor by the same exponential expression.

Ex. 1. $a^{-2}/c^2 \equiv c^{-2}/a^2$; $a^{-3}/b^{-2} \equiv b^2/a^3$.

Ex. 2. $\dfrac{a^{-1}x^2}{c^{-2}y^3} \equiv \dfrac{c^2x^2}{ay^3} \equiv c^2x^2a^{-1}y^{-3} \equiv \dfrac{1}{c^{-2}x^{-2}ay^3}$.

Exercise 122.

Find the value of each of the following expressions:

1. 4^{-2}.
2. 5^{-3}.
3. $(3/4)^{-1}$.
4. $(2\frac{1}{2})^{-2}$.
5. $(2/3)^{-3}$.
6. $1/5^{-3}$.
7. $1/.3^{-4}$.
8. 35^0.
9. $8^{\frac{4}{3}}$.
10. $4^{-\frac{5}{2}}$.
11. $9^{-\frac{3}{2}}$.
12. $(4/9)^{-\frac{1}{2}}$.
13. $(27/8)^{\frac{4}{3}}$.
14. $(81/16)^{-\frac{5}{4}}$.
15. $(-27)^{\frac{5}{3}}$.
16. $(-125)^{-\frac{2}{3}}$.
17. $8^{-\frac{2}{3}} \cdot 4^{\frac{1}{2}}$.
18. $(1/64)^{-\frac{2}{3}} \cdot (1/9)^{\frac{1}{2}}$.
19. $(1/25)^{-\frac{1}{2}} \cdot 27^{-\frac{4}{3}}$.

Write each of the following expressions without using fractional or negative exponents:

20. $a^{\frac{1}{2}}$.
21. $x^{\frac{3}{4}}$.
22. $3x^{\frac{2}{3}}$.
23. $2x^{\frac{1}{2}}y^{-\frac{3}{2}}$.
24. $2ax^{-\frac{5}{6}}$.
25. $(a/b)^{\frac{4}{5}}$.
26. $(x/y)^{\frac{5}{6}}$.
27. $(x/y)^{-\frac{m}{n}}$.
28. $\dfrac{1}{6a^{-2}x^{-\frac{1}{2}}}$.
29. $\dfrac{5a^{-3}x^4}{7y^2z^{-5}}$.
30. $\dfrac{5x^{-1}y^{-3}z^2}{7c^{-2}db^{-\frac{2}{3}}}$.
31. $\dfrac{x^{-3}y^{-2}}{a^{-2}b^{-4}}$.
32. $\dfrac{x^{-1}y^{-2}z^{-3}}{ab^{-1}c^{-2}}$.
33. $\dfrac{b(x-y)^{-1}}{(x+y)^{-2}}$.
34. $\dfrac{a^{-m}b^{-n}}{x^{-c}y^{-s}z^{-r}}$.
35. $\dfrac{(a+b)^{-1}x^{-7}y}{(a+b)^{-2}}$.
36. $\dfrac{x^{-n}(x+y)^{-3}}{a^{-n}(x+y)^{-2}}$.

37. Write each of the expressions in examples 28 to 36 in the integral form, *i.e.*, transfer all the factors from each denominator to the numerator.

38. Using fractional exponents, write $\sqrt{a^3}$; $\sqrt[3]{a^7}$; $\sqrt[5]{x^7}$; $\sqrt[4]{y^5}$.

39. Using negative fractional exponents, write $\sqrt{1/x^3}$; $\sqrt[3]{1/a^5}$; $\sqrt[5]{1/x^7}$; $\sqrt[n]{1/a^m}$.

344. The meaning of any real commensurable exponent having been determined, it remains to prove that any such exponent obeys *all* the laws of positive integral exponents.

For convenience in stating these laws we shall enlarge the meaning of the word *power* so as to include whatever is denoted by an exponent.

345. Exponents were so defined as to obey the following fundamental law:

The product of the **m***th and the* **n***th power of any base is equal to the* $(m+n)$*th power of that base.*

That is, $$a^m a^n \equiv a^{m+n}, \qquad \text{I}$$

where m and n denote any commensurable real numbers.

Ex. 1. $x^{\frac{3}{4}} \cdot x^{\frac{2}{4}} \equiv x^{\frac{3}{4}+\frac{2}{4}} \equiv x^{\frac{5}{4}}$.

Ex. 2. $x^{-\frac{1}{5}} \cdot x^{-\frac{3}{5}} \equiv x^{-\frac{1}{5}+(-\frac{3}{5})} \equiv x^{-\frac{4}{5}}$.

Ex. 3. $x^2a^{-\frac{3}{4}} \cdot x^{-3}a^{\frac{1}{4}} \equiv x^{-1}a^{-\frac{2}{4}} \equiv 1/(xa^{\frac{2}{4}})$.

346. *The quotient of the* **m***th power of any base divided by the* **n***th power of the same base is equal to the* $(m-n)$*th power of that base.*

That is, $$a^m/a^n \equiv a^{m-n}, \qquad \text{II}$$

where m and n denote any commensurable numbers.

Proof. $a^m \div a^n \equiv a^m \times a^{-n} \equiv a^{m-n}$. §§ 342, 345

Ex. 1. $x^{-3} \div x^{-4} \equiv x^{-3-(-4)} \equiv x$. §§ 342, 345

Ex. 2. $a^{\frac{4}{5}} \div a^{-\frac{2}{5}} \equiv a^{\frac{4}{5}-(-\frac{2}{5})} \equiv a^{\frac{6}{5}}$.

Ex. 3. $a^{-\frac{3}{2}} \div a^{-\frac{7}{2}} \equiv a^{-\frac{3}{2}-(-\frac{7}{2})} \equiv a^2$.

347. By § 251, $\sqrt[s]{a^r} \equiv \sqrt[sn]{a^{rn}}$; $\therefore\ a^{\frac{r}{s}} \equiv a^{\frac{rn}{sn}}$.

Hence $$a^{\frac{r}{s}+\frac{p}{q}} \equiv a^{\frac{rq}{sq}+\frac{ps}{sq}} \equiv a^{\frac{rq+ps}{sq}}.$$

Ex. 1. $x^{\frac{3}{4}} \cdot x^{\frac{2}{3}} \equiv x^{\frac{3}{4}+\frac{2}{3}} \equiv x^{\frac{9+8}{12}} \equiv x^{\frac{17}{12}}$.

Ex. 2. $x^{\frac{3}{4}}/x^{\frac{2}{3}} \equiv x^{\frac{3}{4}-\frac{2}{3}} \equiv x^{\frac{9-8}{12}} \equiv x^{\frac{1}{12}}$.

Exercise 123.

Simplify each of the following expressions:

1. x^7x^0.	**5.** $x^6x^5x^{-7}$.	**9.** $a^3b^{-2} \times a^{-4}b^3$.
2. $a^{-5}a^5$.	**6.** $a^{-5}a^{-2}a^8$.	**10.** $3\,x^{-\frac{3}{4}}y^{\frac{1}{3}} \times 2\,x^{-\frac{1}{4}}y^{\frac{4}{3}}$.
3. $a^{-4}a^5$.	**7.** $x^{\frac{3}{4}}x^{\frac{2}{4}}$.	**11.** $4\,x^{-\frac{2}{7}}a^{-\frac{2}{5}} \times 3\,x^{\frac{9}{7}}a^{\frac{7}{5}}$.
4. $x^{-2}x^{-3}$.	**8.** $x^{-\frac{7}{4}} \cdot x^{\frac{5}{4}}$.	**12.** $2\,x^{\frac{4}{9}}b^{\frac{7}{11}} \times 5\,x^{-\frac{7}{9}}b^{-\frac{9}{11}}$.

13. a/a^{-2}.	**15.** $4^{-2}/4^{-3}$.	**17.** x^{-4}/x^{-7}.	**19.** $x^{-\frac{3}{4}}/x^{-\frac{5}{4}}$.
14. x^0/x^{-3}.	**6.** $7^{-3}/7^{-4}$.	**18.** x^{-7}/x^{-11}.	**20.** $y^{\frac{2}{5}}/y^{-\frac{3}{5}}$.

21. $(3\,x^{-4})/(6\,x^{-7})$.	**29.** $x^{-\frac{3}{2}}/x^{-\frac{5}{6}}$.
22. x^{m-n}/x^{-m}.	**30.** $x^{\frac{3}{2}}/x^{-\frac{3}{8}}$.
23. x^{-1}/x^{n-2}.	**31.** $a^{\frac{3}{2}}x^{\frac{3}{4}} \cdot a^{\frac{1}{3}}x^{\frac{2}{3}}$.
24. x^{5-n}/x^{2-n}.	**32.** $a^{-\frac{3}{4}}x^{-\frac{3}{2}} \cdot a^3x^{\frac{3}{4}}$.
25. x^{3n}/x^{2-n}.	**33.** $x^{\frac{3}{2}}y^{\frac{2}{3}}/(x^{-\frac{1}{3}}y^{-\frac{1}{2}})$.
26. x^{-n}/x^{3-2n}.	**34.** $\sqrt[5]{a^{-3}} \div \sqrt[3]{a^{-8}}$.
27. $x^{\frac{3}{4}} \cdot x^{-\frac{2}{3}}$.	**35.** $\sqrt[3]{x^m} \cdot \sqrt[4]{x^m} \div \sqrt[12]{x^m}$.
28. $a^{\frac{2}{3}}a^{\frac{1}{2}}a^{-\frac{7}{6}}$.	

348. If m and n denote any commensurable real numbers, the other three laws of exponents are:

The **n***th power of the* **m***th power of any base is equal to the* **mn***th power of that base.*

That is, $$(a^m)^n \equiv a^{mn}. \qquad \text{III}$$

The **n***th power of the product of any number of factors is equal to the product of the* **n***th powers of the factors.*

That is, $$(ab\cdots)^n \equiv a^nb^n\cdots. \qquad \text{IV}$$

The n*th power of the quotient of one number by another is equal to the quotient of the* n*th power of the first by the* n*th power of the second.*

That is, $$(a/b)^n \equiv a^n/b^n. \qquad \text{V}$$

Ex. 1. $(x^2)^{-3} \equiv x^{2(-3)} \equiv x^{-6} \equiv 1/x^6.$ By III

Ex. 2. $(8^{-2})^{-\frac{2}{3}} = 8^{-2\left(-\frac{2}{3}\right)} = 8^{\frac{4}{3}} = 16.$ By III

Ex. 3. $(a^{-\frac{3}{5}})^{-\frac{7}{4}} \equiv a^{-\frac{3}{5}\left(-\frac{7}{4}\right)} \equiv a^{\frac{21}{20}}.$ By III

Ex. 4. $(x^{-1}y^{-3})^{-2} \equiv (x^{-1})^{-2}(y^{-3})^{-2} \equiv x^2y^6.$ By III, IV

Ex. 5. $(9^{-\frac{3}{4}}x^{-6})^{-\frac{2}{3}} \equiv (9^{-\frac{3}{4}})^{-\frac{2}{3}}(x^{-6})^{-\frac{2}{3}} \equiv 9^{\frac{1}{2}}x^4 \equiv 3x^4.$

Ex. 6. $(x^2/a^{-3})^{-2} \equiv (x^2a^3)^{-2} \equiv x^{-4}a^{-6}.$

Ex. 7. $(x^{\frac{2}{3}}/y^{\frac{1}{2}})^{-6} \equiv (x^{\frac{2}{3}}y^{-\frac{1}{2}})^{-6} \equiv x^{-4}y^3.$

Ex. 8. $$\left(\frac{4x^{-2}}{9y^3}\right)^{-\frac{3}{2}} \equiv \left(\frac{2x^{-1}}{3y^{\frac{3}{2}}}\right)^{-3} \equiv \frac{2^{-3}x^3}{3^{-3}y^{-\frac{9}{2}}}$$
$$\equiv 27\,y^{\frac{9}{2}}x^3/8.$$

Ex. 9. $$\frac{a^{\frac{2}{3}}\sqrt{b^{-1}}}{b\sqrt[3]{a^{-2}}} \div \sqrt{\frac{a\sqrt{b^{-4}}}{b\sqrt{a^{-2}}}} \equiv \frac{a^{\frac{2}{3}}b^{-\frac{1}{2}}}{ba^{-\frac{2}{3}}} \div \left(\frac{ab^{-2}}{ba^{-1}}\right)^{\frac{1}{2}}$$
$$\equiv a^{\frac{4}{3}}b^{-\frac{3}{2}} \div (a^2b^{-3})^{\frac{1}{2}}$$
$$\equiv a^{\frac{4}{3}}b^{-\frac{3}{2}} \div ab^{-\frac{3}{2}} \equiv a^{\frac{1}{3}}.$$

349. *Proof of laws* III, IV, V *when* m *and* n *are positive fractions.*

Let p, q, r, and s denote any positive integers.

To prove III,
$$(a^{\frac{p}{q}})^{\frac{r}{s}} \equiv (\sqrt[s]{\sqrt[q]{a^p}})^r \qquad \S\ 337$$
$$\equiv (\sqrt[qs]{a^p})^r \qquad \S\ 227$$
$$\equiv [(\sqrt[qs]{a})^p]^r \qquad \S\ 226$$
$$\equiv (\sqrt[qs]{a})^{pr} \equiv a^{\frac{pr}{qs}}. \qquad \text{III}$$

To prove IV, $(ab)^{\frac{r}{s}} \equiv \sqrt[s]{(ab)^r}$ § 337

$\equiv \sqrt[s]{a^r b^r}$ § 119

$\equiv \sqrt[s]{a^r} \cdot \sqrt[s]{b^r}$ § 224

$\equiv a^{\frac{r}{s}} b^{\frac{r}{s}}.$ IV

To prove V, $(a/b)^{\frac{r}{s}} \equiv \sqrt[s]{(a/b)^r}$ § 337

$\equiv \sqrt[s]{a^r/b^r}$ § 186

$\equiv \sqrt[s]{a^r}/\sqrt[s]{b^r}$ § 225

$\equiv a^{\frac{r}{s}}/b^{\frac{r}{s}}.$ V

350. *Proof of laws* III, IV, V *when* ***m*** *and* ***n*** *are negative.* Let h and k denote any positive integers or fractions.

To prove III, $(a^{-h})^{-k} \equiv 1 \div (1/a^h)^k$ § 339

$\equiv 1 \div (1/a^{hk})$ §§ 186, 118

$\equiv a^{hk}.$ III

To prove IV, $(ab)^{-h} \equiv 1/(ab)^h$ § 339

$\equiv 1/a^h b^h$ § 119

$\equiv (1/a^h)(1/b^h) \equiv a^{-h}b^{-h}.$ IV

To prove V, $(a/b)^{-h} \equiv 1 \div (a/b)^h$ § 339

$\equiv 1 \div (a^h/b^h)$ § 186

$\equiv b^h/a^h \equiv a^{-h}/b^{-h}.$ V

The verification of laws III, IV, and V when m or n is zero is left as an exercise for the pupil.

Exercise 124.

Simplify each of the following expressions:

1. $(a^{-2})^3.$
2. $(a^2)^{-3}.$
3. $(x^{\frac{2}{3}})^{-6}.$
4. $(x^{-\frac{2}{5}})^4.$
5. $(x^{-\frac{2}{3}})^{-\frac{3}{4}}.$
6. $(a^n)^{-3}$
7. $(a^{-n})^{-4}.$
8. $(a^{-\frac{r}{s}})^{\frac{m}{n}}.$
9. $(\sqrt[3]{x^{-2}})^4.$

10. $(\sqrt[5]{a^{-2}})^{-\frac{5}{6}}$.

11. $(\sqrt[r]{a^{-s}})^{-4}$.

12. $(\sqrt[r]{a^{-n}})^{-m}$.

13. $(x^{\frac{3}{2}}a^{-2})^{-3}$.

14. $(4\,a^{-1})^{-\frac{3}{2}}$.

15. $(8\,a^{-9})^{\frac{1}{3}}$.

16. $(x^{-2}y^{-1})^{-5}$.

17. $(8\,a^{-\frac{3}{2}}x^{-1})^{-\frac{5}{3}}$.

18. $(x^{-\frac{1}{8}}a^{\frac{1}{2}})^{-12}$.

19. $(a^{-2}c^{-\frac{3}{4}})^{-8}$.

20. $(a^{-8}/32)^{-\frac{1}{4}}$.

21. $(2\,a^{-1}\sqrt{x})^{-3}$.

22. $(3\,x^2\sqrt[3]{a^2})^{-9}$.

23. $(x^{-n}c^{-m})^{-\frac{r}{s}}$.

24. $(3\sqrt{x}\sqrt[3]{c^{-1}})^{-6}$.

25. $(a^2/x^{-3})^{-4}$.

26. $[-2^3\,a^{-4}/(4\,x^{-1})]^{-2}$.

27. $(32\,a^{-\frac{1}{2}}/x^3)^{-\frac{2}{5}}$.

28. $\sqrt[5]{a^2b^{10}}/\sqrt[3]{ab^6}$.

29. $\sqrt{a^3b^{-2}}/\sqrt[3]{a^{-4}b^5}$.

30. $\sqrt[3]{x^{-2}y^{-1}}/\sqrt{x^{-8}y^{-4}}$.

31. $\sqrt{a^3b^{-2}}/\sqrt[3]{a^{-4}b^5}$.

32. $\sqrt[4]{(a+b)^6}\cdot(a^2-b^2)^{-\frac{1}{2}}$.

33. $\sqrt[3]{x^{-1}\sqrt{y^3}}/\sqrt{y\sqrt[3]{x}}$.

34. $\sqrt[3]{(a+b)^5}\cdot(a+b)^{-\frac{2}{3}}$.

35. $\left(\dfrac{a^{-2}b}{a^3b^{-4}}\right)^{-3} \div \left(\dfrac{ab^{-1}}{a^{-3}b^2}\right)^5$.

36. $\left\{\dfrac{\sqrt[3]{a}}{\sqrt[4]{b^{-1}}}\cdot\left(\dfrac{b^{\frac{1}{4}}}{a^{\frac{1}{3}}}\right)^2 \div \dfrac{a^{-\frac{1}{3}}}{b^{-\frac{1}{2}}}\right\}^6$.

37. $\left(\dfrac{x^{-2}y^3}{x^3y^{-2}}\right)^{-\frac{1}{5}}\cdot\left(\dfrac{y^3x^{-3}}{x^3y^{-3}}\right)^{-1}$.

38. $\left(\dfrac{a^{-3}}{b^{-\frac{2}{3}}}\right)^{-\frac{3}{2}} \div \left(\dfrac{a^{-\frac{1}{4}}b^{\frac{1}{2}}}{a^2c^{-1}}\right)^{-2}$.

39. $\left(\dfrac{a^{-\frac{2}{3}}x^{\frac{1}{2}}}{x^{-1}a}\right)^2 \div \sqrt[3]{\dfrac{a^{-1}}{x^{-3}}}$.

351. The following examples are applications of the methods of multiplication, division, and evolution, to polynomials whose terms involve fractional and negative exponents:

Ex. 1. Multiply $\sqrt[3]{a}+1+1/\sqrt[3]{a}$ by $\sqrt[3]{a}+1/\sqrt[3]{a}-1$.

The terms $+1$ and -1 may be regarded as the coefficients of a^0; hence arranging both expressions in descending powers of a, we have

$$\begin{array}{lllll}
a^{\frac{1}{3}} + 1 + a^{-\frac{1}{3}} & & & & \\
\underline{a^{\frac{1}{3}} - 1 + a^{-\frac{1}{3}}} & & & & \\
a^{\frac{2}{3}} + a^{\frac{1}{3}} + 1 & & & & \\
\quad\;\; - a^{\frac{1}{3}} - 1 - a^{-\frac{1}{3}} & & & & \\
\qquad\qquad\;\; \underline{+ 1 + a^{-\frac{1}{3}} + a^{-\frac{2}{3}}} & & & & \\
a^{\frac{2}{3}} \qquad\;\; + 1 \qquad\quad + a^{-\frac{2}{3}} & & & &
\end{array}$$

Ex. 2. Divide $16\,a^{-3} + 5\,a^{-1} - 6\,a^{-2} + 6$ by $1 + 2\,a^{-1}$.

Arrange in ascending powers of a.

$$
\begin{array}{rrrr|l}
16\,a^{-3} & -\ 6\,a^{-2} & +\ 5\,a^{-1} & +\ 6 & 2\,a^{-1} + 1 \\
\cline{5-5}
\underline{16\,a^{-3}} & \underline{+\ 8\,a^{-2}} & & & 8\,a^{-2} - 7\,a^{-1} + 6 \\
 & -\ 14\,a^{-2} & +\ 5\,a^{-1} & & \\
 & \underline{-\ 14\,a^{-2}} & \underline{-\ 7\,a^{-1}} & & \\
 & & 12\,a^{-1} & +\ 6 & \\
 & & \underline{12\,a^{-1}} & \underline{+\ 6} &
\end{array}
$$

Ex. 3. Find the square root of

$$4\,x + 2\,x^{\frac{7}{6}} - 4\,x^{\frac{5}{6}} - 4\,x^{\frac{4}{3}} + x^{\frac{5}{3}} + x^{\frac{2}{3}}.$$

Arrange in descending powers of x.

$$
\begin{array}{r|l|l}
 & x^{\frac{5}{3}} - 4\,x^{\frac{4}{3}} + 2\,x^{\frac{7}{6}} + 4\,x - 4\,x^{\frac{5}{6}} + x^{\frac{2}{3}} & x^{\frac{5}{6}} - 2\,x^{\frac{3}{6}} + x^{\frac{1}{3}} \\
 & x^{\frac{5}{3}} & \\
\hline
2\,x^{\frac{5}{6}} - 2\,x^{\frac{3}{6}} & -\ 4\,x^{\frac{4}{3}} + 2\,x^{\frac{7}{6}} + 4\,x & \\
 & -\ 4\,x^{\frac{4}{3}} \qquad\quad + 4\,x & \\
\hline
2\,x^{\frac{5}{6}} - 4\,x^{\frac{3}{6}} + x^{\frac{1}{3}} & 2\,x^{\frac{7}{6}} \qquad - 4\,x^{\frac{5}{6}} + x^{\frac{2}{3}} & \\
 & 2\,x^{\frac{7}{6}} \qquad - 4\,x^{\frac{5}{6}} + x^{\frac{2}{3}} & \\
\hline
\end{array}
$$

Exercise 125.

Multiply:

1. $a^{\frac{4}{3}} + a^{\frac{2}{3}}b^{\frac{2}{3}} + b^{\frac{4}{3}}$ by $a^{\frac{2}{3}} - b^{\frac{2}{3}}$.

2. $x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$ by $x^{\frac{1}{3}} + y^{\frac{1}{3}}$.

3. $3\,x^{\frac{1}{3}} - 5 + 8\,x^{-\frac{1}{3}}$ by $4\,x^{\frac{1}{3}} + 3\,x^{-\frac{1}{3}}$.

4. $3\,a^{\frac{3}{5}} - 4\,a^{\frac{1}{5}} - a^{-\frac{1}{5}}$ by $3\,a^{\frac{1}{5}} + a^{-\frac{1}{5}} - 6\,a^{-\frac{3}{5}}$.

5. $a^{\frac{4}{3}} + a^{\frac{2}{3}}b^{\frac{2}{3}} + b^{\frac{4}{3}}$ by $a^{\frac{4}{3}} - a^{\frac{2}{3}}b^{\frac{2}{3}} + b^{\frac{4}{3}}$.

6. $x^{\frac{7}{4}} - x^{\frac{3}{2}} + x^{\frac{5}{4}} - x$ by $x^{\frac{3}{4}} + x^{\frac{1}{2}}$.

7. $x^{\frac{5}{4}} - x^{\frac{3}{4}} + x^{\frac{1}{4}} - x^{-\frac{1}{4}}$ by $x^{\frac{3}{4}} + x^{\frac{1}{4}}$.

8. $a^{\frac{2}{3}} + b^{\frac{2}{3}} + c^{\frac{2}{3}} - b^{\frac{1}{3}}c^{\frac{1}{3}} - c^{\frac{1}{3}}a^{\frac{1}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}}$ by $a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}}$.

9. $x^n + x^{\frac{n}{2}} + 1$ by $x^{-n} + x^{-\frac{n}{2}} + 1$.

10. $\frac{1}{8}a^{\frac{3}{2}} - \frac{1}{12}ab^{\frac{1}{2}} + \frac{1}{18}a^{\frac{1}{2}}b - \frac{1}{27}b^{\frac{3}{2}}$ by $\frac{1}{2}a^{\frac{1}{2}} + \frac{1}{3}b^{\frac{1}{2}}$.

11. $c^x + 2c^{-x} - 7$ by $5 - 3c^{-x} + 2c^x$.

Divide:

12. $21x + x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1$ by $3x^{\frac{1}{3}} + 1$.

13. $15a - 3a^{\frac{1}{3}} - 2a^{-\frac{1}{3}} + 8a^{-1}$ by $5a^{\frac{2}{3}} + 4$.

14. $5b^{\frac{2}{3}} - 6b^{\frac{1}{3}} - 4b^{-\frac{2}{3}} - 4b^{-\frac{1}{3}} - 5$ by $b^{\frac{1}{6}} - 2b^{-\frac{1}{6}}$.

15. $21a^{3x} + 20 - 27a^x - 26a^{2x}$ by $3a^x - 5$.

16. $x^{\frac{3}{4}}y^{-\frac{3}{4}} + 2 + x^{-\frac{3}{4}}y^{\frac{3}{4}}$ by $x^{\frac{1}{4}}y^{-\frac{1}{4}} - 1 + x^{-\frac{1}{4}}y^{\frac{1}{4}}$.

17. $a^{\frac{5}{2}} + a^2b^{\frac{1}{3}} - a^{\frac{3}{2}}b^{\frac{2}{3}} - ab + a^{\frac{1}{2}}b^{\frac{4}{3}} + b^{\frac{5}{3}}$ by $a^{\frac{1}{2}} + b^{\frac{1}{3}}$.

18. $x^{\frac{7}{3}}y^{-\frac{14}{5}} + y^{\frac{14}{5}}x^{-\frac{7}{3}}$ by $x^{\frac{1}{3}}y^{-\frac{2}{5}} + y^{\frac{2}{5}}x^{-\frac{1}{3}}$.

19. $a^{\frac{4}{3}} - 2 + a^{-\frac{4}{3}}$ by $a^{\frac{2}{3}} - a^{-\frac{2}{3}}$.

20. $8c^{-n} - 8c^n + 5c^{3n} - 3c^{-3n}$ by $5c^n - 3c^{-n}$.

Find the square root of the following expressions:

21. $25a^{\frac{4}{3}} + 16 - 30a - 24a^{\frac{1}{3}} + 49a^{\frac{2}{3}}$.

22. $9x - 12x^{\frac{1}{2}} + 10 - 4x^{-\frac{1}{2}} + x^{-1}$.

23. $4x^2a^{-2} + 12xa^{-1} + 25 + 24x^{-1}a + 16x^{-2}a^2$.

24. $25x^2y^{-2} + \frac{1}{4}y^2x^{-2} - 20xy^{-1} - 2yx^{-1} + 9$.

25. $x^{\frac{8}{5}} - 2a^{-\frac{3}{5}}x^{\frac{11}{5}} + 2a^{\frac{4}{5}}x^{\frac{4}{5}} + a^{-\frac{6}{5}}x^{\frac{14}{5}} - 2a^{\frac{1}{5}}x^{\frac{7}{5}} + a^{\frac{8}{5}}$.

26. $4x^n + 9x^{-n} + 28 - 24x^{-\frac{1}{2}n} - 16x^{\frac{1}{2}n}$.

27. $9x^{-4} - 18x^{-3}\sqrt{y} + 15y \div x^2 - 6\sqrt{y^3} \div x + y^2$.

352. The following examples are applications of the formulas for products and quotients in Chapter IX., to binomials whose terms involve fractional and negative exponents:

(1) $$(2a^{\frac{1}{2}} - x^{-\frac{1}{2}})^3 = (2a^{\frac{1}{2}})^3 + 3(2a^{\frac{1}{2}})^2(-x^{-\frac{1}{2}}) + 3(2a^{\frac{1}{2}})(-x^{-\frac{1}{2}})^2 + (-a^{-\frac{1}{2}})^3. \quad (1)$$

$$= 8a^{\frac{3}{2}} - 12ax^{-\frac{1}{2}} + 6a^{\frac{1}{2}}x^{-1} - a^{-\frac{3}{2}}. \quad (2)$$

(2) $$(x^{\frac{3}{2}} + y^{\frac{3}{2}})(x^{\frac{3}{2}} - y^{\frac{3}{2}}) = (x^{\frac{3}{2}})^2 - (y^{\frac{3}{2}})^2 = x^3 - y^3. \qquad \S\ 122$$

(3) $$(7x - 9y^{-1})(7x + 9y^{-1}) = 49x^2 - 81y^{-2}.$$

(4) $$(4x - 5x^{-1})(4x + 3x^{-1}) = 16x^2 + (3x^{-1} - 5x^{-1})4x - 15x^{-2} \qquad \S\ 123$$

$$= 16x^2 - 8 - 15x^{-2}.$$

(5) $$(x^{\frac{5}{3}} - 1) \div (x^{\frac{1}{3}} - 1) = [(x^{\frac{1}{3}})^5 - 1^5] \div (x^{\frac{1}{3}} - 1)$$

$$= (x^{\frac{1}{3}})^4 + (x^{\frac{1}{3}})^3 + (x^{\frac{1}{3}})^2 + x^{\frac{1}{3}} + 1 \qquad \S\ 129$$

$$= x^{\frac{4}{3}} + x + x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1.$$

(6) $$(x^{\frac{3}{2}} + 27) \div (x^{\frac{1}{2}} + 3) = [(x^{\frac{1}{2}})^3 + 3^3] \div (x^{\frac{1}{2}} + 3)$$

$$= x - 3x^{\frac{1}{2}} + 9.$$

Exercise 126.

Write the value of the following expressions:

1. $(a^{-\frac{3}{4}} + b^{\frac{2}{3}})^2$.
2. $(x^{\frac{4}{5}} - y^{-\frac{2}{3}})^2$.
3. $(m^{\frac{1}{2}} + n^{\frac{2}{3}})^3$.
4. $(c^2 - b^{\frac{2}{3}})^3$.
5. $(r^{-\frac{1}{3}} - s^{-\frac{1}{2}})^3$.
6. $(2a^{\frac{1}{2}} - a^{-\frac{1}{2}})^3$.
7. $(r^{-2} + b^{\frac{1}{2}})^4$.
8. $(r^2 - 3n^{-\frac{1}{2}})^4$.
9. $(\frac{1}{2}\sqrt{a} - \frac{1}{3}\sqrt[3]{b})^4$.
10. $(a^{-\frac{3}{5}} - 2b^2c^{\frac{1}{3}})^4$.
11. $(a^{\frac{2}{3}} + b^{-\frac{3}{4}})^5$.
12. $(x^{\frac{1}{2}} - y^{-\frac{2}{3}})^6$.
13. $(x^{\frac{1}{4}} + 1)(x^{\frac{1}{4}} - 1)$.
14. $(x^{\frac{1}{3}} + y^{\frac{1}{3}})(x^{\frac{1}{3}} - y^{\frac{1}{3}})$.
15. $(4x^{\frac{3}{2}} + 3a^{-\frac{1}{2}})(4x^{\frac{3}{2}} - 3a^{-\frac{1}{2}})$.
16. $(3x - 5a^{-1})(3x + 2a^{-1})$.

17. $(ab - c^{\frac{1}{2}})(ab + 5\,x^{-1})$.

18. $(x - 9\,a) \div (x^{\frac{1}{2}} + 3\,a^{\frac{1}{2}})$.

19. $(a^{-2x} - 16) \div (a^{-x} - 4)$.

20. $(x^{-3a} + 8) \div (x^{-a} + 2)$.

21. $(c^{2x} - c^{-x}) \div (c^{x} - c^{-\frac{1}{2}x})$.

22. $(1 - 8\,a^{-3}) \div (1 - 2\,a^{-1})$.

23. $(x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1)(x^{\frac{1}{3}} - 1)$.

24. $(x^{-4} - 1) \div (x^{-1} + 1)$.

25. $(x^{5n} + 32) \div (x^{n} + 2)$.

26. $(x^{3} - y^{3}) \div (x^{\frac{3}{2}} - y^{\frac{3}{2}})$.

27. $(x^{2} + y^{2}) \div (x^{\frac{2}{3}} + y^{\frac{2}{3}})$.

28. $(x - 243\,y^{\frac{5}{3}}) \div (x^{\frac{1}{5}} - 3\,y^{\frac{1}{3}})$.

CHAPTER XXVII

INDETERMINATE EQUATIONS AND SYSTEMS

353. Division by zero. As a quotient, $0/0$ denotes the number which multiplied by 0 is equal to 0 (§ 85). By § 74 *any* number multiplied by 0 is equal to 0; hence $0/0$ *denotes any number whatever, or is indeterminate.* That is, *when the dividend is zero, division by zero is indeterminate.*

As a quotient, $a/0$ denotes the number which multiplied by zero is equal to a. But any number, however large, multiplied by zero, is zero; hence the division of a by 0 is impossible. That is, *when the dividend is not zero, division by zero is impossible in the sense that no number can express the quotient or any part of it.*

354. The forms $0/0$ and $a/0$. As an answer to a problem the *indeterminate form* $0/0$ denotes that the problem is indeterminate, *i.e.*, has an unlimited number of answers.

As an answer to a problem, $a/0$ denotes that the problem involves inconsistent conditions, and is therefore impossible, as is illustrated by the following problem:

Prob. A and B are travelling in the direction PR at the rates of a and b miles per hour. At 12 o'clock A is at P and B at Q, which is c miles to the right of P. Find when they are together.

$P \qquad\qquad\qquad Q \qquad\qquad\qquad\qquad\qquad R$

Let distances measured to the right from P, and periods of time after 12 o'clock, be regarded as positive.

Let $x =$ the number of hours from 12 o'clock to the time when A and B are together.

Then $$ax = bx + c. \tag{1}$$

Hence $$x = \frac{c}{a-b}. \tag{2}$$

Discussion. If $c > 0$ and $a > b$, x is positive; that is, A will overtake B at some time after 12 o'clock.

If $c > 0$ and $a < b$, x is negative; that is, A and B were together at some time *before* 12 o'clock.

If $c = 0$ and $a \neq b$, $x = 0$; that is, A and B are together at 12 o'clock, but not before or after that time.

If $c = 0$ and $a = b$, $x = 0/0$; that is, A and B are always together under the conditions; and the problem is indeterminate, *i.e.*, has an unlimited number of answers.

If $c \neq 0$ and $a = b$, $x = c/0$; that is, A and B can never be together as they are always at a fixed distance apart; the problem involves inconsistent conditions, and is therefore impossible.

Observe that the fraction $\dfrac{c}{a-b}$ assumes the form $\dfrac{0}{0}$ by reason of *two independent conditions;* namely, $c = 0$ and $a = b$. In any such case the form $0/0$ indicates that the given fraction can have *any* value under the conditions.

355. An **impossible equation** is one which expresses a condition which cannot be satisfied.

E.g., $3x + 5 = 3x - 8$ is an impossible equation; for it expresses the condition $0 \cdot x = -13$, which no value of x can satisfy.

Again, $\sqrt{x} = -3$ is an impossible equation, when $\sqrt{x}$ is restricted to its principal value.

An **impossible system of equations** is a system whose equations are inconsistent (§ 206).

E.g., the system

$$\left.\begin{aligned} ax + by &= c, \quad (1)\\ 3ax + 3by &= 5c, \quad (2)\end{aligned}\right\} \quad (a)$$

is impossible; for its equations are evidently inconsistent (§ 206).

An impossible equation or system of equations is often a particular case of a more general equation or system, in which the solutions involve the form $a/0$.

Thus, the equation $ax = b$ becomes impossible only when $a = 0$, and then its root b/a becomes $b/0$.

It will be seen in § 356 that a system of two linear equations in x and y becomes impossible only for a certain relation between the coefficients of its equations, which makes the values of x and y assume the form $a/0$.

Again the system

$$\left.\begin{array}{lr} x + y = 9, & (1) \\ 2x + y = 13, & (2) \\ x + 5y = 16, & (3) \end{array}\right\} (b)$$

is impossible; for the only solution common to (1) and (2) is 4, 5, and this reduces (3) to $29 = 16$.

Equation (3) cannot be obtained from (1) or (2), or by combining (1) and (2); hence it is independent of them *separately* and *jointly*.

System (b) illustrates the principle that

When the number of independent equations in a system exceeds the number of unknowns, the system is impossible.

356. A **defective system** is one which lacks one or more of the full number of solutions which we would expect from the degrees of its equations.

E.g., the system

$$\left.\begin{array}{lr} a^2x^2 - b^2y^2 = c^2, & (1) \\ ax - (b + e)y = c, & (2) \end{array}\right\} (a)$$

which has, in general, two solutions (§ 306), becomes defective when

$$e = 0.$$

For, dividing (1) by (2) when $e = 0$, we obtain

$$ax + by = c. \qquad (3)$$

Equations (2) and (3) form a system equivalent to system (a); hence system (a) has but one solution when $e = 0$.

357. An **indeterminate equation** is one which has an *unlimited* number of solutions. Thus any equation in two or more unknowns is indeterminate.

An **indeterminate system** of equations is one which has an *unlimited* number of solutions.

E.g., the system

$$\left.\begin{array}{r} 3x + 4y + 5z = 0, \\ x - y - 2z = 0, \end{array}\right\} (a)$$

is an indeterminate system; for, assigning any value whatever to z, we can find a corresponding set of values of x and y. Hence, system (a) has an unlimited number of solutions, and is indeterminate.

Again, the system

$$\left.\begin{aligned} 2x + 3y - z &= 15, & (1)\\ 3x - y + 2z &= 8, & (2)\\ 5x + 2y + z &= 23, & (3)\end{aligned}\right\}$$

is indeterminate. No two of its equations are equivalent, but any one of them can be obtained from the other two; thus, by adding (1) and (2), we obtain (3). Hence the system contains but two independent equations, and therefore any solution of two of them will be a solution of the third.

These examples illustrate the following principle:

When the number of independent equations in a system is less than the number of unknowns, the system is indeterminate.

Ex. By discussing its solution, show that the system

$$\left.\begin{aligned} ax + by &= c,\\ a'x + b'y &= c',\end{aligned}\right\} \quad (a)$$

is (i) indeterminate if $\qquad \dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{c}{c'};$ (1)

and (ii) impossible if $\qquad \dfrac{a}{a'} = \dfrac{b}{b'} \neq \dfrac{c}{c'}.$ (2)

By § 207 the values of x and y in system (a) are

$$x = \frac{b'c - bc'}{ab' - a'b}, \quad y = \frac{ac' - a'c}{ab' - a'b}. \tag{3}$$

(i) When condition (1) is satisfied, from (1) we have

$$ab' - a'b = 0, \;\; b'c - bc' = 0, \;\; ac' - a'c = 0;$$

hence the values of x and y in (3) each assume the form $0/0$; that is, the system has an unlimited number of solutions, and is therefore indeterminate.

(ii) When condition (2) is satisfied, we have $ab' - a'b = 0$.

But neither $b'c - bc'$ nor $ac' - a'c$ is zero.

Hence the value of each x and y assumes the form $a/0$; that is, the system has no solution, and is therefore impossible.

The equations in (a) are evidently *equivalent* when (1) is satisfied, and *inconsistent* when (2) is satisfied.

358. Sometimes it is required to find the *positive integral* solutions of an indeterminate equation or system.

The following examples will illustrate the simplest general method of finding such solutions.

Ex. 1. Solve $7x + 12y = 220$ in positive integers.

Dividing by 7, the smaller coefficient, expressing improper fractions as mixed numbers, and adding the proper fractions, we obtain

$$x + y + \frac{5y-3}{7} = 31. \tag{1}$$

Since x and y are integers, $31 - x - y$ is an integer; hence the fraction in (1) denotes an integer.

Multiplying this fraction by such a number as will make the coefficient of y divisible by the denominator with remainder 1 (which in this case is 3), we have

$$\frac{15y-9}{7} = 2y - 1 + \frac{y-2}{7} = \text{an integer.}$$

Hence $$\frac{y-2}{7} = \text{an integer} = p, \text{ suppose.}$$

$$\therefore\ y = 7p + 2. \tag{2}$$

From (1) and (2), $$x = 28 - 12p. \tag{3}$$

Since x and y are *positive* integers, from (2) it follows that $p > -1$, and from (3) it follows that $p < 3$; hence

$$p = 0, 1, 2. \tag{4}$$

From (2), (3), and (4), we obtain the three solutions

$$x = 28,\ 16,\ \ 4;$$

$$y = \ \ 2,\ \ 9,\ 16.$$

Ex. 2. Solve in positive integers the system

$$\left.\begin{aligned} x + y + z &= 43, & (1)\\ 10x + 5y + 2z &= 229. & (2) \end{aligned}\right\}$$

Eliminate z, $8x + 3y = 143$,

or $$y + 2x + \frac{2x - 2}{3} = 47. \qquad (3)$$

$$\therefore \frac{4x - 4}{3} = x - 1 + \frac{x - 1}{3} = \text{an integer.}$$

$$\therefore \frac{x - 1}{3} = \text{an integer} = p, \text{ suppose.}$$

$$\therefore x = 3p + 1. \qquad (4)$$

From (3) and (4), $y = 45 - 8p$. (5)

From (1), (4), and (5), $z = 5p - 3$. (6)

From (6), $p > 0$; and from (5), $p < 6$; hence

$$p = 1, \quad 2, \quad 3, \quad 4, \quad 5.$$

Whence
$$x = 4, \quad 7, \quad 10, \quad 13, \quad 16;$$
$$y = 37, \quad 29, \quad 21, \quad 13, \quad 5;$$
$$z = 2, \quad 7, \quad 12, \quad 17, \quad 22.$$

Thus, the system has five positive integral solutions.

Exercise 127.

Solve in positive integers:

1. $3x + 29y = 151$.
2. $3x + 8y = 103$.
3. $7x + 12y = 152$.
4. $13x + 7y = 408$.
5. $23x + 25y = 915$.
6. $13x + 11y = 414$.
7. $\left.\begin{array}{l} 6x + 7y + 4z = 122, \\ 11x + 8y - 6z = 145. \end{array}\right\}$
8. $\left.\begin{array}{l} 12x - 11y + 4z = 22, \\ -4x + 5y + z = 17. \end{array}\right\}$
9. $\left.\begin{array}{l} 20x - 21y = 38, \\ 3y + 4z = 34. \end{array}\right\}$
10. $5x - 14y = 11$.
11. $\left.\begin{array}{l} 13x + 11z = 103, \\ 7z - 5y = 4. \end{array}\right\}$
12. $14x - 11y = 29$.

13. A farmer buys horses at \$ 111 a head, cows at \$ 69, and spends \$ 2256. How many of each does he buy?

14. A drover buys sheep at \$ 4 a head, pigs at \$ 2, and oxen at \$ 17. If 40 animals cost him \$ 301, how many of each kind does he buy?

15. I have 27 coins, which are dollars, half-dollars, and dimes, and they amount to \$ 9.80. How many of each sort have I?

16. A drover buys sheep at \$ 3.50 a head, turkeys at \$ $1.33\frac{1}{3}$, and hens at \$ 0.50. If 100 animals cost him \$ 100, how many of each does he buy?

CHAPTER XXVIII

THEORY OF LIMITS

359. A **variable** is a quantity which is, or is conceived to be, continually changing in value.

E.g., the time since any past event is a variable; so also is the height of an ascending or a descending balloon.

The amount of water in a cistern which is being filled by a continuous stream is a variable; and the *number* which measures this amount is a *variable number.*

Variable numbers are usually represented by the final letters of the alphabet, as x, y, z.

A **constant** is a quantity whose value is fixed or invariable. *Constant numbers* are usually represented by figures or the first letters of the alphabet, as 4, 7, a, b, c.

E.g., the time between any two past events is a constant; and the *number* which measures this time is a constant number.

360. Limit of a variable. When, according to its law of change, a variable approaches indefinitely near, and continually nearer a constant, but can never reach it, the variable is said *to approach the constant as its* **limit.**

E.g., let A, B, and N be three *fixed* points in the straight line AN; then AB and NB will be *constant distances.*

Suppose a point P, starting from A, moves 1/2 the distance from A to B, or to C, the first second; 1/2 the remaining distance, or to D, the next second; 1/2 the remaining distance, or to E, the next second; and so on indefinitely; then AP, NP, and BP will be *variable distances.*

The *variable distance* AP will approach indefinitely near and continually nearer the *constant distance* AB, but can never reach it; that is, the variable distance AP will approach AB as its *limit*. The *variable distance* NP will approach indefinitely near and continually nearer the *constant distance* NB, but can never reach it; that is, the variable distance NP will approach NB, as its *limit*. The variable distance BP will approach indefinitely near and continually nearer zero, but can never reach it; that is, BP will approach zero as its *limit*.

The variable AP will always be less than its limit AB; and the variable NP will always be greater than its limit NB.

Again, suppose the number of sides of a regular polygon inscribed in a given circle is increased from 4 to 8; from 8 to 16; from 16 to 32, and so on indefinitely; then the area of the inscribed polygon will approach the area of the circle as its limit, and the area between the perimeter of the polygon and the circumference of the circle will approach zero as its limit.

Some variables change according to such laws that they approach limits, others do not. The theory of limits applies only to such variables as approach limits.

361. Notation. The sign $\doteq$ is read 'approaches as a limit'; thus $x \doteq a$ is read 'x approaches a as its limit.'

The phrase 'as a limit' is sometimes omitted, $x \doteq a$ being read 'x approaches a.'

The expression $\text{lt}(x)$ is read 'the limit of x.'

Thus, $\text{lt}(x) = a$, read 'the limit of x is equal to a,' is only another way of writing $x \doteq a$.

362. *The difference between a variable and its limit approaches zero as its limit.* That is, if $\boldsymbol{x} \doteq \boldsymbol{a}$, then $\boldsymbol{x} - \boldsymbol{a} \doteq 0$.

Conversely, *if the difference between a variable and a constant approaches zero as its limit, the variable approaches the constant as its limit.* That is, if $\boldsymbol{x} - \boldsymbol{a} \doteq \mathbf{0}$, then $\boldsymbol{x} \doteq \boldsymbol{a}$.

Proof. $x - a$ approaches just as near to 0 as x does to a and no nearer; hence if $x \doteq a$, then $x - a \doteq 0$; and conversely.

363. *A variable cannot approach two unequal limits at the same time.*

Proof. In approaching as a limit the more remote of two unequal constants, a variable would evidently reach a value between the two constants, and thereafter while it approached the one as a limit, it would recede from the other, which therefore is not a limit. Hence the theorem.

364. *The limit of the sum of a constant and a variable is the sum of the constant and the limit of the variable.*

That is, if $\boldsymbol{x \doteq a}$, $\boldsymbol{c + x \doteq c + a}$.

Proof. The sum $c + x$ approaches just as near to $c + a$ as x does to a, and no nearer; hence if $x \doteq a$, $c + x \doteq c + a$.

365. *The limit of the product of a constant and a variable is the product of the constant and the limit of the variable.*

That is, if $\boldsymbol{x \doteq a}$, $\boldsymbol{cx \doteq ca}$, when $c \neq 0$.

Proof. By § 362, if $x \doteq a$, $x - a \doteq 0$.

Choose a constant k as small as you please; then $x - a$ will become arithmetically less than k/c.

Hence $cx - ca$ will become arithmetically less than k.

But $cx - ca$ cannot become 0, since $x - a$ cannot.

Hence when $x \doteq a$, $cx \doteq ca$.

366. *If two variables are always equal and one approaches a limit, the other approaches the same limit.*

That is, if $\boldsymbol{y = x}$ and $\boldsymbol{x \doteq a}$, then $\boldsymbol{y \doteq a}$.

Proof. If $y = x$, then y approaches just as near to a as x does, and no nearer; hence, if $x \doteq a$, $y \doteq a$.

367. *If two variables are always equal and each approaches a limit, their limits are equal.*

That is, if $\boldsymbol{y = x}$, and $\boldsymbol{x \doteq a}$, and $\boldsymbol{y \doteq b}$, then $\boldsymbol{b = a}$.

Proof. If $y = x$ and $x \doteq a$, then, by § 366, $y \doteq a$. (1)

But by hypothesis, $y \doteq b$. (2)

By (1), (2), and § 363, $b = a$.

368. *If* $\text{lt}(v) = 0$ *and* $\text{lt}(w) = 0$, *then* $\text{lt}(vw) = 0$.

Proof. If $v \doteq 0$ and $w \doteq 0$, vw approaches nearer to 0 than either v or w; but vw cannot equal 0, since neither v nor w can.

Hence if $v \doteq 0$ and $w \doteq 0$, $vw \doteq 0$.

369. *The limit of the variable sum of two or more variables is the sum of their limits.*

That is, if $\quad x \doteq a,\ y \doteq b,\ z \doteq c, \cdots,$

then $$x + y + z + \cdots \doteq a + b + c + \cdots.$$

Proof. Let $v_1 = x - a,\ v_2 = y - b,\ v_3 = z - c, \cdots;$

then $(x + y + z + \cdots) - (a + b + c + \cdots) = v_1 + v_2 + v_3 + \cdots.$

Now, however small a constant k may be, each one of the n variables $v_1, v_2, v_3, \cdots$ can become arithmetically less than k/n; hence their sum can become arithmetically less than k. But, since $x + y + z + \cdots$ is variable, $v_1 + v_2 + v_3 + \cdots$ cannot reach and remain zero. Hence $v_1 + v_2 + v_3 + \cdots \doteq 0$.

Hence $\quad x + y + z + \cdots \doteq a + b + c + \cdots.$ § 362

370. *The limit of the variable product of two or more variables is the product of their limits.*

That is, $\text{lt}(xyz \cdots) = \text{lt}(x) \cdot \text{lt}(y) \cdot \text{lt}(z) \cdots.$

Proof. Let $\quad x \doteq a,\ y \doteq b,$

and let $\quad v = x - a,\ w = y - b;$

then $\quad x = a + v,\ y = b + w.$

Hence $\quad xy = ab + aw + bv + vw.$

$$\therefore \ \operatorname{lt}(xy) = ab + \operatorname{lt}(aw + bv + vw) \qquad §§ 367, 364$$
$$= ab + a\operatorname{lt}(w) + b\operatorname{lt}(v) + \operatorname{lt}(vw) \qquad §§ 369, 365$$
$$= ab \qquad § 368$$
$$= \operatorname{lt}(x)\cdot\operatorname{lt}(y). \qquad (1)$$

By (1), $\quad \operatorname{lt}(xy\cdot zu) = \operatorname{lt}(xy)\cdot\operatorname{lt}(zu)$

$$= \operatorname{lt}(x)\cdot\operatorname{lt}(y)\cdot\operatorname{lt}(z)\cdot\operatorname{lt}(u).$$

And so on for any number of variables.

371. *The limit of the variable quotient of two variables is the quotient of their limits, when the limit of the divisor is not zero.*

That is, $\operatorname{lt}(\boldsymbol{x}/\boldsymbol{y}) = \operatorname{lt}(\boldsymbol{x})/\operatorname{lt}(\boldsymbol{y})$, *when* $\operatorname{lt}(\boldsymbol{y}) \neq \mathbf{0}$.

Proof. Let $\quad z = x/y$, or $x = yz$;

then $\quad \operatorname{lt}(z) = \operatorname{lt}(x/y), \qquad (1)$

and $\quad \operatorname{lt}(x) = \operatorname{lt}(yz) = \operatorname{lt}(y)\cdot\operatorname{lt}(z). \qquad (2)$

Dividing (2) by $\operatorname{lt}(y)$ when $\operatorname{lt}(y) \neq 0$, we obtain

$$\operatorname{lt}(z) = \operatorname{lt}(x)/\operatorname{lt}(y), \text{ when } \operatorname{lt}(y) \neq 0. \qquad (3)$$

Equating the two values of $\operatorname{lt}(z)$ in (1) and (3), we have

$$\operatorname{lt}(x/y) = \operatorname{lt}(x)/\operatorname{lt}(y), \text{ when } \operatorname{lt}(y) \neq 0.$$

Ex. $$\operatorname{lt}\left(\frac{xyz}{cvw}\right) = \frac{\operatorname{lt}(xyz)}{\operatorname{lt}(cvw)} \qquad § 371$$
$$= \frac{\operatorname{lt}(x)\cdot\operatorname{lt}(y)\cdot\operatorname{lt}(z)}{c\cdot\operatorname{lt}(v)\cdot\operatorname{lt}(w)}. \qquad § 370$$

372. $\operatorname{Lt}(\boldsymbol{x}^n) \equiv [\operatorname{lt}(\boldsymbol{x})]^n$, *where* $\boldsymbol{n}$ *is a positive integer.*

Proof. $\quad \operatorname{lt}(x^n) \equiv \operatorname{lt}(x\cdot x\cdot x \cdots \text{ to } n \text{ factors}) \qquad$ by notation

$$\equiv \operatorname{lt}(x)\cdot\operatorname{lt}(x)\cdots \text{ to } n \text{ factors} \qquad § 370$$
$$\equiv [\operatorname{lt}(x)]^n. \qquad \text{by notation}$$

E.g., if $x \doteq a$, $\operatorname{lt}(x^4) = a^4$, $\operatorname{lt}(x^6) = a^6$, $\operatorname{lt}(x^{11}) = a^{11}$.

Exercise 128.

If $x \doteq a,\ y \doteq b,\ z \doteq c,\ u \doteq e,\ v \doteq 0,$ find:

1. $\text{lt}\,(ax)$.
2. $\text{lt}\,(cx + av)$.
3. $\text{lt}\,(xy - 7\,zu)$.
4. $\text{lt}\,(x^2 + x^3)$.
5. $\text{lt}\,(x^3 - x^4)$.
6. $\text{lt}\,(xy^3 - zu^5)$.
7. $\text{lt}\,(x^3/y^5)$.
8. $\text{lt}\,(2\,x^5/y^8)$.
9. $\text{lt}\,(vx^3y/z^4)$.
10. $\text{lt}\,(x^2y^4 + mx^3z^5 + nv)$.
11. $\text{lt}\left(\dfrac{x^2 + my^3 + nv}{xz^4 - mx^4 - ny^5}\right)$.
12. $\text{lt}\,(x^2y^3 + mxz^2 + nxv^2)$.
13. $\text{lt}\left(\dfrac{x^4y^2}{z^5} + \dfrac{mz^2}{ny^7}\right)$.

373. *When the quotient of two variables or the product or sum of two or more variables is equal to a constant, the quotient, product, or sum of their limits is equal to the same constant.*

Proof. (i) Let $xy = m$; then $xyz = mz$.

We multiply by z to make the members variable.

$\therefore\ \text{lt}\,(x)\cdot\text{lt}\,(y)\cdot\text{lt}\cdot(z) = m\cdot\text{lt}\,(z)$. §§ 370, 365

Divide by $\text{lt}\,(z)$, $\text{lt}\,(x)\cdot\text{lt}\,(y) = m$.

(ii) Let $x \div y = m$; then $x = my$.

$$\therefore\ \text{lt}\,(x) = \text{lt}\,(my) = m\cdot\text{lt}\,(y).$$

$$\therefore\ \text{lt}\,(x) \div \text{lt}\,(y) = m, \text{ when } \text{lt}\,(y) \neq 0.$$

(iii) Let $x + y + z + \cdots = m.$ (1)

$$y + z + \cdots = m - x.$$

$$\therefore\ \text{lt}\,(y) + \text{lt}\,(z) + \cdots = m - \text{lt}\,(x).$$

$$\therefore\ \text{lt}\,(x) + \text{lt}\,(y) + \text{lt}\,(z)\cdots = m.$$

374. $\text{Lt}\,(c/y) \equiv c/\text{lt}\,(y)$, *when* $\text{lt}\,(y) \neq 0$.

Proof. Let $z = c/y$; then $zy = c$.

Hence $\text{lt}\,(z) = \text{lt}\,(c/y)$, (1)

and $\text{lt}\,(z)\cdot\text{lt}\,(y) = c$. § 370

Hence $\text{lt}\,(z) = c/\text{lt}\,(y)$, when $\text{lt}\,(y) \neq 0$. (2)

From (1) and (2), $\text{lt}\,(c/y) = c/\text{lt}\,(y)$, when $\text{lt}\,(y) \neq 0$.

375. $\text{Lt}(x^{\frac{m}{n}}) \equiv [\text{lt}(x)]^{\frac{m}{n}}$.

Proof. Let $$z = x^{\frac{m}{n}}; \text{ then } z^n = x^m. \quad (1)$$

From (1), $$\text{lt}(z) = \text{lt}(x^{\frac{m}{n}}), \quad (2)$$

and $$[\text{lt}(z)]^n = [\text{lt}(x)]^m. \quad \S\S\ 367,\ 372$$

Hence $$\text{lt}(z) = [\text{lt}(x)]^{\frac{m}{n}}. \quad (3)$$

From (2), (3), $$\text{lt}(x^{\frac{m}{n}}) \equiv [\text{lt}(x)]^{\frac{m}{n}}.$$

Ex. $$\text{lt}(x^{\frac{2}{3}}) \equiv [\text{lt}(x)]^{\frac{2}{3}} = a^{\frac{2}{3}} \text{ when } x \doteq a.$$

376. An **infinitesimal** is a variable whose limit is zero.

Thus, the difference between a variable and its limit is an infinitesimal.

In approaching its limit zero, an infinitesimal becomes indefinitely small and continually smaller, but it never equals zero. A small quantity *becomes* an infinitesimal when it *begins* to approach zero as its limit rather than when it reaches any particular degree of smallness. A quantity, however small, which does not approach zero as its limit is not an infinitesimal.

377. An **infinite** is a variable which under its law of change can exceed any constant however great.

Thus, the reciprocal of an infinitesimal is an infinite.

E.g., if $x \doteq 0$, $1/x$ can exceed any constant number however great; thus, since $1/(0.1^n) \equiv 10^n$, we have

when $$x = .1,\ .1^{10},\ .1^{100},\ .1^{1000},\ .1^{10000},\ \cdots,$$

$$1/x = 10,\ 10^{10},\ 10^{100},\ 10^{1000},\ 10^{10000},\ \cdots.$$

The general symbol for an arithmetic infinite is ∞; and $x = \infty$ is read 'x increases without limit,' or 'x is infinite.' A *positive infinite* is denoted by $+\infty$, and a *negative infinite* by $-\infty$, read '∞ is a negative infinite.'

An infinite does not approach a limit, but *increases* arithmetically *without limit.*

378. Any number which is neither an infinitesimal nor an infinite is called a **finite number.** All the numbers considered prior to this chapter are finite numbers.

379. Any finite number, not zero, divided by an infinitesimal is an infinite; and conversely, any finite number, not zero, divided by an infinite is an infinitesimal.

That is, when $x \doteq 0$, $a/x = \infty$ (where $a \neq 0$).

And conversely, when $x = \infty$, $a/x \doteq 0$.

Ex. Find $\text{lt}\left(\dfrac{4x^3 - 3x^2 + 5}{7x^3 + 4x - 8}\right)$ when $x = \infty$.

$$\frac{4x^3 - 3x^2 + 5}{7x^3 + 4x - 8} \equiv \frac{4 - 3/x + 5/x^3}{7 + 4/x^2 - 8/x^3}. \qquad \S\ 173$$

$$\therefore\ \text{lt}\left(\frac{4x^3 - 3x^2 + 5}{7x^3 + 4x - 8}\right) \equiv \frac{\text{lt}\,(4 - 3/x + 5/x^3)}{\text{lt}\,(7 + 4/x^2 - 8/x^3)}. \qquad \S\S\ 367, 371$$

$$= 4/7. \qquad \S\S\ 364, 369, 379$$

Exercise 129.

Find the limit of each of the following expressions, when $x = \infty$:

1. $\dfrac{7x^2 - 3x}{5x^2 + 9}$.

2. $\dfrac{ax^3 - bx + e}{mx^3 + cx^2 + nx}$.

3. $\dfrac{(3x^2 - 1)^2}{x^4 + 9}$.

4. $\dfrac{(3x - 4)(5x + 4)}{9x^2 + 8x - 11}$.

5. $\dfrac{(3 + 2x^2)(2x - 7)}{(5x^2 + 7)(7 + 9x)}$.

6. $\dfrac{(3 + 2x^3)(x - 5)}{(4x^3 - 9)(1 + x)}$.

If $x \doteq a$, $y \doteq b$, $z \doteq c$, find the value of:

7. abc, if $xyz = m$.

8. ab/c, if $xy/z = n$.

9. a^3/c^2, if $x^3/z^2 = h$.

10. $\text{Lt}\,(m/x^3 + n/y^2)$.

11. $\text{Lt}\,(n/x^n + h/y^n)$.

12. $\text{Lt}\,(a^n/x^n + b^m/y^m)$.

13. $\text{Lt}\,(y^{\frac{3}{4}})$. 14. $\text{Lt}\,(x^{\frac{4}{5}}y^{\frac{1}{2}})$. 15. $\text{Lt}\,(\sqrt{xyz})$.

16. $\text{Lt}\left(\dfrac{x^2y^{\frac{2}{3}} - mz^{\frac{4}{5}}}{xy^{\frac{3}{4}} - nz^{\frac{3}{2}}}\right)$. 17. $\text{Lt}\left(\dfrac{\sqrt{xy}}{z^{\frac{2}{3}}} - \dfrac{m\sqrt[3]{xy^2}}{n\sqrt{xz^3}}\right)$.

380. Fractions which assume the form 0/0.

Substituting 1 for x in the fraction $(x^2-1)/(x-1)$, we have

$$\frac{x^2-1}{x-1} = \frac{1-1}{1-1} = \frac{0}{0}.$$

That is, by the method in § 12 we fail to obtain any *definite* value for the fraction $(x^2-1)/(x-1)$, when $x = 1$.

But for any value of x other than 1, we have

$$(x^2-1)/(x-1) \equiv x + 1. \tag{1}$$

Hence (1) holds true when $x \doteq 1$.

$$\therefore \lim_{x \doteq 1}\left(\frac{x^2-1}{x-1}\right) \equiv \lim_{x \doteq 1}(x+1) = 2. \tag{2}$$

That is, 2 is the limit which the fraction approaches when $x \doteq 1$.

The first member of (2) is read 'the limit of $(x^2-1)/(x-1)$, when $x \doteq 1$.'

The example above suggests the following definition:

The **value of an expression** for any particular value of its variable is the *limit* which the expression approaches when the variable approaches this particular value as its limit.

This definition applies to any expression, but we shall use it only when the simpler one in § 12 fails.

Ex. Find the value of $(x^2+x-2)/(x^2-1)$, when $x = 1$.

Putting 1 for x in this fraction, we obtain the form $0/0$.

Hence to find the value of this fraction, when $x = 1$, we must find its limit when $x \doteq 1$.

For values of x other than 1, we have

$$\frac{x^2+x-2}{x^2-1} \equiv \frac{(x-1)(x+2)}{(x-1)(x+1)} \equiv \frac{x+2}{x+1}.$$

$$\therefore \lim_{x \doteq 1}\left(\frac{x^2+x-2}{x^2-1}\right) \equiv \lim_{x \doteq 1}\left(\frac{x+2}{x+1}\right)$$

$$\equiv \frac{\text{lt}\,(x+2)}{\text{lt}\,(x+1)} = \frac{3}{2}.$$

Hence the value of the fraction when $x = 1$ is $3/2$.

Note that we cannot apply § 371 to the given fraction, for the limit of its divisor is 0 when $x \doteq 1$.

Observe that the indeterminate form $0/0$ arises here by reason of *one condition;* viz. $x =$ some particular value. In any such case an indeterminate form simply indicates that the method of *evaluation by substitution* (§ 12) fails, and that the more general method *by limits* must be used.

Exercise 130.

Find the value of each of the following expressions:

1. $\dfrac{x^3-1}{x-1}$, when $x=1$.

2. $\dfrac{x^3+1}{x^2-1}$, when $x=-1$.

3. $\dfrac{x^2-5x+6}{x^2-4}$, when $x=2$.

4. $\dfrac{x^2+7x+12}{x^2-9}$, when $x=-3$.

5. $\dfrac{x^5-1}{x^3-1}$, when $x=1$.

6. $\dfrac{x^3-a^3}{x^2-a^2}$, when $x=a$.

7. $\dfrac{x^4-a^4}{x^3-a^3}$, when $x=a$.

8. $\dfrac{x^5+a^5}{x^3+a^3}$, when $x=-a$.

9. $\dfrac{(x^2-a^2)^{\frac{1}{3}}}{(x-a)^{\frac{1}{3}}}$, when $x=a$.

10. $\dfrac{x^2-ax-3x-3a}{x^2-a^2}$, when $x=-a$.

381. $a/0$, or absolute infinity. The expression $a/0$, read 'a by zero,' frequently occurs in mathematics, and the question arises 'what does it mean?' By § 353, $a/0$ must symbolize that of which no part can be expressed by any number however large; hence it symbolizes that which transcends all number, or *absolute infinity*, of which we can have no positive idea.

The expression $a/0$ is commonly denoted by the symbol ∞. When this notation is adopted, this meaning of ∞ must be clearly distinguished from that in § 377, where ∞ denotes an infinite, or a variable which increases without limit.

In this book ∞ never denotes $a/0$.

382. Certain combinations of 0 and $a/0$; as

$$(a/0)\div(a/0),\ (a/0)\cdot 0,\ a/0-a/0, \text{ etc.},$$

produce **additional indeterminate forms**. But any expression which assumes any one of these forms can be reduced to an identical expression which for the same values of its variables will assume the fundamental form $0/0$.

E.g., we have the identities,

$$(a/x)\div(a/y)\equiv y/x \tag{1}$$

$$(a/x)\cdot y\equiv ay/x \tag{2}$$

$$a/x-a/y\equiv a(y-x)/(xy). \tag{3}$$

If $x=0$ and $y=0$, (1), (2), and (3) become

$$(a/0)\div(a/0)\equiv 0/0$$

$$(a/0)\cdot 0\equiv 0/0$$

$$a/0-a/0\equiv 0/0.$$

If in the identity $$x^{a-y}\equiv x^a/x^y$$

we put $x=0$ and $y=a$, we obtain

$$0^0\equiv 0/0.$$

That is, 0^0 is an indeterminate form.

LAWS OF INCOMMENSURABLE NUMBERS

383. The laws, already proved for commensurable numbers, are, by the theory of limits, easily proved for incommensurable numbers.

384. Proof of the fundamental laws.

Let a, b, c be any incommensurable constant numbers, and let x, y, z be commensurable variable numbers such that $x \doteq a,\ y \doteq b,\ z \doteq c$.

Proof of (*A*). $$x+y=y+x. \qquad \S\ 36$$

$$\therefore\ \operatorname{lt}(x)+\operatorname{lt}(y)=\operatorname{lt}(y)+\operatorname{lt}(x); \qquad \S\S\ 367,\ 369$$

that is, $$a+b=b+a.$$

Proof of (A'). $$xy = yx. \qquad \S\ 49$$

$$\therefore\ \mathrm{lt}(x)\,\mathrm{lt}(y) = \mathrm{lt}(y)\,\mathrm{lt}(x); \qquad \S\S\ 367,\ 370$$

that is, $$a \cdot b = b \cdot a.$$

Proof of (C). $$(x + y)z = xz + yz. \qquad \S\ 60$$

$$\therefore\ [\mathrm{lt}(x) + \mathrm{lt}(y)]\,\mathrm{lt}(z) = \mathrm{lt}(x)\,\mathrm{lt}(z) + \mathrm{lt}(y)\,\mathrm{lt}(z); \qquad \S\S\ 369,\ 370$$

that is, $$(a + b)c = ac + bc.$$

Laws (B), (B'), and (C') follow from laws (A), (A'), and (C), as in §§ 36, 49, and 88.

385. *If* x *is commensurable and* $x \doteq 0$, *then* $a^x \doteq 1$, *when* $a \neq 1$ *or* 0.

E.g., giving to x the successive values $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\cdots$, and finding the corresponding values of 16^x, we obtain the results below:

When $x = \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \frac{1}{32},\ \frac{1}{64},\ \frac{1}{128},\ \frac{1}{256}\cdots,$

$16^x = 4,\ 2,\ 1.4,\ 1.19,\ 1.09,\ 1.04,\ 1.019,\ 1.009\cdots.$

Observe that each value of 16^x is the square root of the preceding value.

From this table of values it is evident that when $x \doteq 0$, 16^x will approach indefinitely near and continually nearer 1; but it cannot reach 1, since x cannot reach 0, and $16^x = 1$, when, and only when, $x = 0$.

386. Meaning of a^m, m incommensurable.

Let $$x < m < z,$$

where x and z are commensurable, and let $a > 1$; then in harmony with the meaning of *commensurable* exponents we assume a^m to denote a number such that

$$a^x < a^m < a^z. \tag{1}$$

Let $$x \doteq m, \text{ and } z \doteq m;$$

then $$\mathrm{lt}(z - x) \equiv \mathrm{lt}(z) - \mathrm{lt}(x) \equiv 0.$$

Hence by § 385, $$\mathrm{lt}(a^{z-x}) \equiv 1. \tag{2}$$

Now $$a^z - a^x \equiv a^x(a^{z-x} - 1).$$

$$\therefore \text{lt}(a^z - a^x) \equiv \text{lt}(a^x)[\text{lt}(a^{z-x}) - 1] \quad \S\S\ 370,\ 364$$

$$\equiv \text{lt}(a^x)(1 - 1) = 0. \quad \text{by (2)}$$

From (1), $$a^z - a^x > a^m - a^x;$$

hence, as $$a^z - a^x \doteq 0,\ a^m - a^x \doteq 0;$$

$$\therefore a^m \equiv \text{lt}(a^x) \text{ when } x \doteq m.$$

That is, *any base* $\boldsymbol{a}$ *with an incommensurable exponent* $\boldsymbol{m}$ *denotes the limit of* $\boldsymbol{a^x}$ *when* $\boldsymbol{x \doteq m}$.

387. Proof of laws of exponents I, II, IV, V.

Let m and n be any incommensurable constant numbers, and let x and y be commensurable variable numbers such that $x \doteq m$, $y \doteq n$.

Proof of I. $$a^x a^y \equiv a^{x+y}. \quad \S\ 345$$

Hence $$\text{lt}(a^x a^y) \equiv \text{lt}(a^{x+y}) \equiv a^{m+n}. \quad (1)$$

But $$\text{lt}(a^x a^y) \equiv \text{lt}(a^x) \cdot \text{lt}(a^y) \equiv a^m a^n. \quad (2)$$

From (1) and (2), $$a^m a^n \equiv a^{m+n}. \quad \text{Law I}$$

Proof of II. By law I we have

$$a^{m-n} a^n \equiv a^m.$$

Hence by § 32 $$a^{m-n} \equiv a^m / a^n. \quad \text{Law II}$$

Proof of IV. $$a^x b^x \equiv (ab)^x. \quad \S\ 348$$

Hence $$\text{lt}(a^x b^x) \equiv \text{lt}[(ab)^x] = (ab)^m. \quad (5)$$

But $$\text{lt}(a^x b^x) \equiv \text{lt}(a^x) \cdot \text{lt}(b^x) \equiv a^m \cdot b^m. \quad (6)$$

From (5) and (6), $$(ab)^m \equiv a^m b^m. \quad \text{Law IV}$$

Proof of V. $$a^x / b^x \equiv (a/b)^x. \quad \S\ 348$$

Hence $$\text{lt}(a^x / b^x) \equiv \text{lt}[(a/b)^x] \equiv (a/b)^m. \quad (7)$$

But $$\text{lt}(a^x / b^x) \equiv \text{lt}(a^x) / \text{lt}(b^x) \equiv a^m / b^m. \quad (8)$$

From (7) and (8), $$(a/b)^m \equiv a^m / b^m. \quad \text{Law V}$$

388. To prove law III for incommensurable exponents we need the following theorem of limits:

If y *and* z *are commensurable and* m *and* c *are incommensurable; then, when* $y \doteq m$ *and* $z \doteq c$, $z^y \doteq c^m$.

Proof. For all values of z (except 0 and 1), and hence for z variable, by § 386, we have

$$z^y - z^m \doteq 0, \text{ when } y \doteq m. \tag{1}$$

By § 375, $$z^m - c^m \doteq 0, \text{ when } z \doteq c. \tag{2}$$

From (1) and (2), $$(z^y - z^m) + (z^m - c^m) \doteq 0;$$

that is, $$z^y - c^m \doteq 0, \text{ when } y \doteq m \text{ and } z \doteq c.$$

Hence $$z^y \doteq c^m \text{ when } y \doteq m \text{ and } z \doteq c. \tag{3}$$

Proof of III. Using the same notation as in § 387, we have

$$(a^x)^y \equiv a^{xy}.$$

Hence $$\text{lt}\,[(a^x)^y] \equiv \text{lt}\,(a^{xy}) \equiv a^{mn}. \tag{4}$$

But by (3), $$\text{lt}\,[(a^x)^y] \equiv [\text{lt}\,(a^x)]^{\text{lt}(y)} = (a^m)^n. \tag{5}$$

From (4) and (5), $$(a^m)^n \equiv a^{mn}.$$

From these laws for incommensurable numbers the other laws follow by the proofs already given for commensurable numbers.

VARIATION.

389. Two variables are often so related that the value of one depends upon the value of the other.

E.g., the distance a train runs at a given speed depends upon the time it runs, and this distance increases when the time increases.

The length of an elastic cord depends upon its tension, and this length varies when the tension varies.

If $y = 5\,x^2$, the value of y depends upon the value of x, and y varies when x varies.

We shall here consider only the simplest kinds of variation.

390. Direct variation. When the ratio of two variables is a constant, either variable is said to *vary directly as* the other.

The symbol $\propto$, read '*varies directly as*,' is called the *symbol of direct variation.* When placed between two variables it denotes that their ratio is some constant.

The word '*directly*' is sometimes omitted.

E.g., $y \propto x$, read 'y varies directly as x,' denotes that $y/x = m$, where m is some constant.

Again, if $y = 3x$, $y/x = 3$; hence, $y \propto x$ or $x \propto y$.

391. *If one variable varies directly as another, either variable is a constant multiple of the other; and conversely.*

Proof. If $\quad y \propto x,\ y/x = m;\ \therefore y = mx$, or $x = (1/m)y$.

Conversely, if $y = mx$, $y/x = m$; $\therefore y \propto x$, or $x \propto y$.

E.g., the area of a rectangle = base into altitude.

Hence if the *altitude* is *constant*, the area $\propto$ the base.

And if the *base* is constant, the area $\propto$ the altitude.

392. If $y \propto x$, and if x', y' and x'', y'' are any two sets of corresponding values of x and y, then

$$y' : x' = y'' : x''. \tag{1}$$

Proof. If $y \propto x$, $y'/x' = m$ and $y''/x'' = m$. (2)

From the equal ratios in (2), we have the proportion (1).

Conversely, if $y' : x' = y'' : x''$, $y = mx$ and $y \propto x$.

393. Inverse variation. One variable is said to *vary inversely as* another when the first varies as the *reciprocal* of the second.

That is, ***y*** *varies inversely as* ***x***, when $\boldsymbol{y \propto 1/x}$.

394. *If one variable varies inversely as another, the product of the two variables is a constant; and conversely.*

Proof. If $\quad y \propto 1/x,\ y = m(1/x);\quad \therefore\ yx = m.$

Conversely, if $yx = m$, $y = m(1/x)$; $\quad \therefore\ y \propto 1/x$.

E.g., if $yx = 3$, y varies inversely as x.

395. Joint variation. One variable is said to *vary as* two others *jointly* when it varies as the product of the two.

That is, ***y*** *varies as* ***x*** *and* ***z*** *jointly, when* $\boldsymbol{y \propto (xz)}$, *or* $\boldsymbol{y = m(xz)}$.

E.g., if $W =$ the amount of work done by M men in D days; then,

if M and D both vary,	$W \propto M \times D$;
if M is constant,	$W \propto D$;
if D is constant,	$W \propto M$.

One variable is said to *vary directly* as a second, and *inversely* as a third, when it varies as the product of the second into the reciprocal of the third.

That is, y varies directly as x, and inversely as z, when

$$y \propto x(1/z), \text{ or } y = mx(1/z).$$

E.g., if $yz = 3x$, $y = 3x(1/z)$; hence $y \propto x(1/z)$.

396. In each of the preceding cases of variation, the value of the constant, m, can be found when any set of corresponding values of the variables is known.

Ex. **1.** Given $y \propto x$, and $y = 6$ when $x = 2$; find the constant ratio of y to x.

Since $y \propto x$, $y = mx$, where m is some constant. (1)

Since $y = 6$, when $x = 2$, from (1) we have

$$6 = 2m, \text{ or } m = 3 = y/x.$$

Ex. **2.** The volume, V, of a pyramid varies jointly as its height, H, and the area of its base, B. When the area of the base is 60 square feet and the height 14 feet, the volume is 280 cubic feet. Find the area of the base of a pyramid whose volume is 390 cubic feet, and whose height is 26 feet.

Since $V \propto BH$, $V = mBH$, where m is some constant. (1)

Substituting the given values of V, B, H, in (1), we have

$$280 = m \times 60 \times 14, \text{ or } m = \tfrac{1}{3}.$$

$$\therefore V = \tfrac{1}{3}BH.$$

Hence, when $V = 390$, and $A = 26$, we have

$$390 = \tfrac{1}{3} B \times 26.$$

$$\therefore\ B = 45, \text{ the number of sq. ft. in base.}$$

397. The simplest way to treat *variations* is to convert them into *equations*.

Ex. If $u \propto y$ and $y \propto x$, prove that $u \propto x$.

Since $u \propto y$ and $y \propto x$, by § 391 we have

$u = ay$ and $y = bx$, where a and b are some constants.

$\therefore\ u = abx. \quad \therefore\ u \propto x.$ § 390

398. *If* $u \propto x$ *when* y *is constant, and* $u \propto y$ *when* x *is constant, then* $u \propto xy$ *when* x *and* y *both vary.*

Let x', y', u' be one set and x'', y'', u'' another set of corresponding values of x, y, z, when all change together.

Let x change from x' to x'', y remaining constant, and suppose that in consequence u changes from u' to u_1; then since $u \propto x$ when y is constant, by § 392 we have

$$u' : x' = u_1 : x''. \tag{1}$$

Now let y change from y' to y'', x remaining constant; then u will change from u_1 to u''; hence as $u \propto y$ when x is constant, we have

$$u_1 : y' = u'' : y''. \tag{2}$$

Multiplying (1) by (2), and dividing the antecedents by u_1, we have

$$u' : x'y' = u'' : x''y''.$$

Hence $u \propto xy$. § 392

Similarly it may be proved that, if u varies as each one of the three variables x, y, z when the other two are constant, then $u \propto xyz$ when they all change; and so on.

E.g., let A denote the area of a triangle, B its base, and H its altitude; then

$A \propto B$, when H is constant,

and $A \propto H$, when B is constant;

hence $A \propto BH$, when B and H both change.

Exercise 131.

1. If x varies directly as y, and $y=7$ when $x=18$; find x when $y=21$.

2. If y varies inversely as x, and $y=4$ when $x=15$; find y when $x=12$.

3. If x varies jointly as y and z, and $x=6$ when $y=3$ and $z=2$; find x when $y=5$, $z=7$.

4. If $x^2 \propto y$ and $z^2 \propto y$, then $xz \propto y$.

5. If $x \propto 1 \div y$, and $y=4$ when $x=15$; find y when $x=6$.

6. If x varies directly as y and inversely as z, and $x=10$ when $y=15$ and $z=6$; find x when $y=8$, $z=2$.

7. If x varies directly as y and inversely as z, and $x=14$ when $y=10$ and $z=14$; find z when $x=49$, $y=45$.

8. If $x \propto 1 \div y$, and $y \propto 1 \div z$, prove $z \propto x$.

9. If $3x+7y \propto 3x+13y$, and $y=3$ when $x=5$; find the equation between x and y.

10. If the cube of x varies as the square of y, and if $x=3$ when $y=5$; find the equation between x and y.

11. If the area of a circle varies as the square of its radius, and if the area of a circle is 154 square feet when the radius is 7 feet; find the area of a circle whose radius is 10 feet 6 inches.

12. The velocity of a falling body varies directly as the time during which it has fallen from rest, and the velocity at the end of 2 seconds is 64. Find the velocity at the end of 5 seconds.

13. The volume of a sphere varies directly as the cube of its radius, and the volume of a sphere whose radius is 1 foot is 4.188 cubic feet. Find the volume of a sphere whose radius is 3 feet.

14. The pressure of a gas varies jointly as its density and its absolute temperature; also when the density is 1 and the temperature 300, the pressure is 15. Find the pressure when the density is 3 and the temperature is 320.

15. The volume of gas varies directly as the absolute temperature and inversely as the pressure. Also when the pressure is 15 and the temperature 280, the volume is 1 cubic foot. Find the volume when the pressure is 20 and the temperature 300.

16. The distance through which a heavy body will fall from rest varies directly as the square of the time, and a body will fall through 144 feet in 3 seconds. Find how far it will fall in 2 seconds.

17. The pressure of wind on a plane surface varies jointly as the area of the surface and the square of the wind's velocity. The pressure on a square foot is 1 pound when the wind is moving at the rate of 15 miles per hour. Find the velocity of the wind when the pressure on a square yard is 16 pounds.

18. The volume of a right circular cone varies jointly as its height and the square of the radius of its base; and the volume of a cone 7 feet high with a base whose radius is 3 feet is 66 cubic feet. Find the volume of a cone 9 feet high with a base whose radius is 14 feet.

CHAPTER XXIX

THE PROGRESSIONS

399. A **series** is a succession of terms whose values are determined by some one law.

A series is said to be **finite** or **infinite** according as the number of its terms is limited or unlimited.

In this chapter we shall consider only the three forms of series which are called the *arithmetic*, the *geometric*, and the *harmonic progressions*.

ARITHMETIC PROGRESSIONS.

400. An **arithmetic progression** (A. P.) is a series in which the *difference* between any term (after the first) and the preceding term is the same throughout the series.

The difference, which can be either positive or negative, is called the **common difference**.

E.g., the series

$$2,\ 5,\ 8,\ 11,\ 14,\ 17,\ 20,\ 23,\ \cdots, \tag{1}$$

and

$$7,\ 5,\ 3,\ 1,\ -1,\ -3,\ -5,\ -7,\ \cdots, \tag{2}$$

are arithmetic progressions.

In series (1) the common difference is 3, and in (2) it is -2.

If in (2) we add -2 to any term, we obtain the next term.

401. The *n*th term. Let d denote the common difference in an A. P., and a the first term; then, by definition,

$$\text{the second term} = a + d,$$

$$\text{the third term} = a + 2d,$$

and

$$\text{the } n\text{th term} = a + (n-1)d. \tag{1}$$

E.g., if the first term of an A. P. is 4, and the common difference is 5,

the ninth term $= 4 + (9 - 1)5 = 44$,

and the twenty-first term $= 4 + (21 - 1)5 = 104$.

Ex. The fourth and fifty-fourth terms of an A. P. are, respectively, 64 and -61. Find the twenty-seventh term.

$$\left.\begin{array}{lll} \text{Here} & 64 = \text{the fourth term} & = a + 3d, \\ \text{and} & -61 = \text{the fifty-fourth term} & = a + 53d. \end{array}\right\} (a)$$

Solving system (a), we find $a = 71\frac{1}{2}$, $d = -5/2$.

$$\therefore \text{ the twenty-seventh term} = a + 26d = 6\tfrac{1}{2}.$$

402. When three numbers, a, b, c, are in A. P., the middle term b is called the **arithmetic mean** of the other two terms a and c.

403. If a, b, c are in A. P., by definition we have

$$b - a = c - b.$$

$$\therefore\ b = (a + c)/2.$$

That is, *the arithmetic mean of any two numbers is half their sum.*

404. All the terms between any two terms of an A. P. may be called the **arithmetic means** of the two terms.

The following example illustrates how any number of arithmetic means can be inserted between any two numbers.

Ex. Insert 9 arithmetic means between 50 and 80.

Since there are 9 arithmetic means, 80 must be the eleventh term, 50 being the first; hence, by definition, we have

$$\text{the eleventh term} = 50 + 10d = 80.$$

Hence $d = 3$, and the required series is

50, 53, 56, 59, 62, 65, 68, 71, 74, 77, 80.

405. Sum of n terms. Let l denote the nth term, and S the sum of n terms of an A. P.; then

$$S = a + (a + d) + (a + 2\,d) + \cdots + (l - d) + l,$$

or $$S = l + (l - d) + (l - 2\,d) + \cdots + (a + d) + a.$$

Adding the corresponding terms, we have

$$2\,S = (a + l) + (a + l) + (a + l) + \cdots \text{ to } n \text{ terms.}$$

$$\therefore\ S = \tfrac{1}{2}n\,(a + l). \qquad (1)$$

From § 401, $$l = a + (n - 1)\,d. \qquad (2)$$

$$\therefore\ S = \tfrac{1}{2}n\{2\,a + (n - 1)\,d\}. \qquad (3)$$

If any three of the five numbers a, d, l, n, S are given, the other two can be found from equations (1) and (2), or from (3) and (2).

Ex. 1. Find the sum of 20 terms of the A. P.

$$-5 - 1 + 3 + 7 + 11 + \cdots.$$

Here $$a = -5,\ d = 4,\ n = 20.$$

$$\therefore\ S = \tfrac{1}{2}n\{2\,a + (n - 1)d\}$$
$$= 10\{-10 + 19 \times 4\}$$
$$= 660.$$

Ex. 2. Find the sum of the first n consecutive odd numbers, 1, 3, 5 ….

Here $$a = 1,\ d = 2,\ n = n.$$

$$\therefore\ S = \tfrac{1}{2}n\{2\,a + (n - 1)\,d\}$$
$$= \tfrac{1}{2}n\{2 + (n - 1)\,2\}$$
$$= n^2.$$

Hence the sum of n consecutive odd numbers, beginning with 1, is n^2.

Ex. 3. The first term of an A. P. is 6, and the sum of 25 terms is 25. Find the common difference.

Here $a = 6$, $S = 25$, $n = 25$; hence from (3) of § 405 we have

$$25 = \tfrac{1}{2} \times 25\{12 + 24\,d\}$$

$$\therefore\ d = -5/12.$$

Ex. 4. How many terms must be taken of the series 11, 12, 13, ⋯ to make 410?

Here $a = 11$, $d = 1$, $S = 410$; hence from (3) of § 405 we have

$$410 = \tfrac{1}{2}\,n\{22 + (n - 1)\}. \qquad (1)$$

$$\therefore\ n = 20, \text{ or } -41.$$

Since the number of terms must be an arithmetic whole number, the number of terms is 20. See § 297.

Ex. 5. How many terms must be taken of the series -16, -15, -14, ⋯ to make -100?

Here $a = -16$, $d = 1$, $S = -100$; hence we have

$$-100 = \tfrac{1}{2}\,n\{-32 + (n - 1)\}.$$

$$\therefore\ n = 8, \text{ or } 25.$$

Hence the number of terms is 8 or 25.

The sum of the 17 terms following the first 8 must therefore be zero. These 17 terms are -8, -7, $-$ ⋯, 7, 8, and their sum is evidently zero.

Exercise 132.

1. Find the twenty-seventh and forty-first terms in the series 5, 11, 17, ⋯.

2. Find the seventeenth and fifty-fourth terms in the series 10, $11\frac{1}{2}$, 13, ⋯.

3. Find the twentieth and thirteenth terms in the series -3, -2, -1, ⋯.

4. If the twelfth term of an A. P. is 15, and the twentieth term is 25, what is the common difference?

5. The seventh term of an A. P. is 5, and the twelfth term is 30. Find the common difference.

6. The first term of an A. P. is 7, and its third term is 13. Find the tenth term.

7. The first term of an A. P. is 20, and its sixth term is 10. Find the twelfth term.

8. The seventh term of an A. P. is 5, and the fifth term is 7. Find the twelfth term.

9. Which term of the series 5, 8, 11, ⋯ is 65?

10. Which term of the series $\frac{7}{6}$, $\frac{4}{3}$, $\frac{3}{2}$, ⋯ is 18?

11. Insert 6 arithmetical means between 8 and 29.

12. Insert 7 arithmetical means between 269 and 295.

13. Insert 15 arithmetical means between 67 and 43.

14. If a, b, c, d are in A. P., prove that $a + d = b + c$.

15. The sum of the second and fifth terms of an A. P. is 32, and the sum of the third and eighth is 48. Find the first term.

16. The sum of the third and fourth terms of an A. P. is 187, and the sum of the seventh and eighth terms is 147. Find the second term.

Find the sum of each of the following series:

17. 5, 9, 13, ⋯ to 19 terms.

18. 1, $2\frac{1}{4}$, $3\frac{1}{2}$, ⋯ to 12 terms.

19. -5, -1, 3, ⋯ to 20 terms.

20. $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$, ⋯ to 7 terms.

21. 10, $\frac{29}{3}$, $\frac{28}{3}$, ⋯ to 7 terms.

22. $\frac{7}{6}$, 1, $\frac{5}{6}$, ⋯ to 15 terms.

How many terms must be taken of:

23. The series 42, 39, 36, ⋯ to make 315?

24. The series 15, 12, 9, ⋯ to make 45?

25. The series -8, -7, -6, ⋯ to make 42?

26. Find the sum of all the numbers between 100 and 500 which are divisible by 3.

27. Find the sum of all the odd numbers between 100 and 200.

28. The sum of 10 terms of an arithmetical series is 145, and the sum of its fourth and ninth terms is 5 times the third term. Determine the series.

29. Divide 80 into 4 parts which are in A. P., and which are such that the product of the first and fourth parts is $\frac{2}{3}$ of the product of the second and third.

30. Find 4 numbers in A. P., such that the sum of their squares shall be 120, and that the product of the first and last shall be less than the product of the other two by 8.

31. If a body falling to the earth descends a feet the first second, $3a$ the second, $5a$ the third, and so on; (1) how far will it fall during the tth second? (2) how far will it fall in t seconds? *Ans.* $(2t-1)a$, at^2.

32. How many strokes does a common clock make in 12 hours?

33. A debt can be discharged in a year by paying \$1 the first week, \$3 the second week, \$5 the third, and so on. Find the last payment and the amount of the debt.

34. One hundred apples are placed on the ground at the distance of a yard from one another. How far will a person travel, who shall bring them, one by one, to a basket, placed at a distance of a yard from the first apple?

35. Two boys A and B set out at the same time, to meet each other, from two places 343 miles apart, their daily journeys being in A. P.; A's common difference being an increase of two miles, and B's a decrease of 5 miles. On the day at the end of which they met, each travelled exactly 20 miles. Find the duration of each journey.

GEOMETRIC PROGRESSIONS.

406. A **geometric progression** (G. P.) is a series in which the ratio of any term (after the first) to the preceding term is the same throughout the series.

This ratio, which can be either positive or negative, is called the **common ratio.**

E.g., the series

$$2, \quad 6, \quad 18, \quad 54, \quad 162, \quad \cdots,$$

$$8, \quad 4, \quad 2, \quad 1, \quad \tfrac{1}{2}, \quad \cdots,$$

or

$$\tfrac{3}{2}, \quad -1, \quad \tfrac{2}{3}, \quad -\tfrac{4}{9}, \quad \tfrac{8}{27}, \quad \cdots,$$

is a geometric progression (G. P.). In the first series, the common ratio is 3; in the second series it is $1/2$; and in the last it is $-2/3$.

If we multiply any term in either series by the common ratio, the product will be the next term of that series.

407. The *n*th term. Let r denote the common ratio, and a the first term of any G. P.; then by definition

$$\text{the second term} = ar,$$

$$\text{the third term} = ar^2,$$

and

$$\text{the } n\text{th term} = ar^{n-1}. \qquad (1)$$

E.g., if the first term of a G. P. is 8, and the common ratio is $1/2$,

$$\text{the fifth term} = 8 \times (1/2)^{5-1} = 1/2,$$

and

$$\text{the ninth term} = 8 \times (1/2)^{9-1} = 1/32.$$

Ex. The sixth term of a G. P. is 156, and the eighth term is 7644. Find the seventh term.

Here $156 = \text{the sixth term} = ar^5, \qquad (1)$

and $7644 = \text{eighth term} = ar^7. \qquad (2)$

Divide (2) by (1), $49 = r^2. \qquad (3)$

$$\therefore r = \pm 7. \qquad (4)$$

But

$$\text{the seventh term} = \text{sixth term} \times r$$

$$= 156\,(\pm 7) = \pm 1092.$$

408. Sum of *n* terms. Let S denote the sum of n terms; then

$$S = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}$$

$$= a(1 + r + r^2 + r^3 + \cdots + r^{n-2} + r^{n-1})$$

$$= a\frac{1 - r^n}{1 - r}. \qquad \S\ 129$$

Hence $$S = \frac{a(1 - r^n)}{1 - r}. \qquad (1)$$

Let l denote the nth term; then from § 407

$$l = ar^{n-1}. \qquad (2)$$

From (1) and (2)

$$S = \frac{a - rl}{1 - r}. \qquad (3)$$

If any three of the five numbers a, l, n, r, s, are known, the other two may be found from equations (1) and (2), or from (2) and (3).

Ex. Sum the series 6, − 18, 54, ⋯ to 6 terms.

Here $a = 6$, $r = -18 \div 6 = -3$, $n = 6$.

From (1),
$$S = \frac{6\{1 - (-3)^6\}}{1 - (-3)}$$

$$= \tfrac{6}{4}\{1 - 3^6\}$$

$$= -1092.$$

409. When three numbers, a, b, c, are in G. P., the middle term b is called the **geometric mean** of the other two terms a and c.

410. If a, b, c are in G. P., by § 406, we have

$$c : b = b : a. \quad \therefore\ b = \sqrt{ac}.$$

That is, *the geometric mean of any two numbers is the mean proportional between them.*

411. All the terms between any two terms of a G. P. may be called the **geometric means** of the two terms.

412. *To insert **m** geometric means between **a** and **b**.*

Calling a the first term, b will be the $(m+2)$th term; hence by (1) of § 407, we have

$$b = ar^{m+1}.$$

$$\therefore\ r = \sqrt[m+1]{b \div a}. \qquad (1)$$

Hence the m means required are ar, ar^2, $\cdots$ ar^m, in which r has the value given in (1).

Ex. Insert 6 geometric means between 56 and $-7/16$.

Here $a = 56,\ b = -7/16,$ and $m + 1 = 7.$

$$\therefore\ r = \sqrt[7]{-\frac{7}{16} \div 56} = \sqrt[7]{-\frac{1}{2^4 \times 2^3}} = -\frac{1}{2}.$$

Hence $r = -1/2$, and the 6 means required are

$-28,\ 14,\ -7,\ 7/2,\ -7/4,\ 7/8.$

Exercise 133.

Find the last term in the following series:

1. $2, 4, 8, \cdots$ to 9 terms.
2. $2, 3, 4\frac{1}{2}, \cdots$ to 6 terms.
3. $3, -3^2, 3^3, \cdots$ to $2n$ terms.
4. $x, 1, 1/x, \cdots$ to 30 terms.
5. The first term of a G. P. is 3, and the third term is 4. Find the fifth term.
6. The third term of a G. P. is 1, and the sixth term is $-1/8$. Find the tenth term.
7. The fourth term of a G. P. is 0.016, and the seventh term is 0.000128. Find the first term.
8. The fourth term of a G. P. is $1/18$, and the seventh term is $-1/486$. Find the sixth term.
9. Insert 3 geometric means between 486 and 6.

10. Insert 4 geometric means between 1/8 and 128.

11. Insert 5 geometric means between 3 and 0.000192.

12. Insert 4 geometric means between a^3b^{-6} and $a^{-2}b^4$.

Find the sum of the following series:

13. 64, 32, 16, $\cdots$ to 10 terms.

14. 8.1, 2.7, 0.9, $\cdots$ to 7 terms.

15. 3, -1, 1/3, $\cdots$ to 6 terms.

16. 1/2, 1/3, 2/9, $\cdots$ to 7 terms.

17. $-2/5$, 1/2, $-5/8$, $\cdots$ to 6 terms.

18. 2, -4, 8, $\cdots$ to $2p$ terms.

413. When $r < 1$ arithmetically, the successive terms of a G. P. become smaller and smaller arithmetically, and the G. P. is said to be a *decreasing progression.*

414. *The limit of the sum of an infinite number of terms of a decreasing G. P. is* $\dfrac{a}{1-r}$.

Proof. From (1) of § 412, we have

$$S = \frac{a}{1-r} - \frac{ar^n}{1-r}. \tag{1}$$

Now if $r < 1$ arithmetically, and the number of terms, or n, is increased without limit, then

$$r^n \doteq 0. \quad \therefore \frac{ar^n}{1-r} \doteq 0.$$

Hence from (1), by § 364, we obtain

$$\text{lt}\,(S) = \frac{a}{1-r}. \tag{2}$$

The *limit of the sum* of an infinite number of terms of a series is often called the *sum of the series.*

E.g., if $a=2$ and $r=1/2$, we have the decreasing G. P.,

$$2,\ 1,\ 1/2,\ 1/4,\ 1/8,\ 1/16,\ 1/32,\ 1/64,\ 1/128.\ \cdots. \qquad (1)$$

The sum of an *infinite* number of terms of this series approaches 4 as its limit. For suppose that we bisect a line four inches long, and take away one of the parts; then bisect the remainder, and take away one of the parts; and continue this process without limit. It is evident that the part remaining will approach zero as its limit, and the sum of the successive parts taken away will approach four inches as its limit. But the numbers of inches in the successive parts taken away will be the terms of series (1). Hence the sum of an infinite number of terms of that series approaches 4 as its limit.

Ex. 1. Find the sum of the series 1, 1/2, 1/4, ⋯.

Here $$a=1,\ r=1/2.$$

From (2), $$\text{lt}\,(S)=\frac{1}{1-1/2}=2.$$

Ex. 2. Find the sum of the series 9, -3, 1, ⋯.

Here $$a=9,\ r=-1/3.$$

From (2), $$\text{lt}\,(S)=\frac{9}{1-(-1/3)}=\frac{27}{4}=6\tfrac{3}{4}.$$

Ex. 3. Express $0.4\dot{2}\dot{3}$ as a common fraction.

$$0.4\dot{2}\dot{3}=0.4232323\cdots=\frac{4}{10}+\frac{23}{10^3}+\frac{23}{10^5}+\cdots.$$

Now, $$\frac{23}{10^3}+\frac{23}{10^5}+\frac{23}{10^7}+\cdots=\frac{23}{10^3}\div\left(1-\frac{1}{10^2}\right)$$

$$=\frac{23}{10^3}\times\frac{10^2}{99}=\frac{23}{990}.$$

$$\therefore\ 0.4\dot{2}\dot{3}=\tfrac{4}{10}+\tfrac{23}{990}=\tfrac{419}{990}.$$

Ex. 4. Find the infinite G. P. whose sum is 18, and whose second term is -8.

Here $$ar=-8, \qquad (1)$$

and $$\frac{a}{1-r}=18. \qquad (2)$$

Divide (1) by (2), $$r(1-r)=-4/9.$$

$$\therefore\ r^2-r-4/9=0.$$

$$\therefore\ r=-1/3 \text{ or } 4/3.$$

Only the value $-1/3$ is admissible for r, since the series is a decreasing one.

From (1), $\quad a = -8 \div (-1/3) = 24.$

Hence the series is $\quad 24,\ -8,\ 8/3,\ -8/9,\ \cdots.$

Exercise 134.

Find the sum of each of the following series:

1. $9,\ 6,\ 4,\ \cdots.$

2. $\frac{1}{2},\ -\frac{1}{4},\ \frac{1}{8},\ \cdots.$

3. $\frac{1}{3},\ \frac{2}{9},\ \frac{4}{27},\ \cdots.$

4. $\frac{8}{5},\ -1,\ \frac{5}{8},\ \cdots.$

5. $0.9,\ 0.03,\ 0.001,\ \cdots.$

6. $0.8,\ -0.4,\ 0.2,\ \cdots.$

Express as a common fraction:

7. $0.\dot{3}$. 8. $0.1\dot{6}$. 9. $0.\dot{2}\dot{4}$. 10. $0.3\dot{7}\dot{8}$. 11. $0.\dot{0}3\dot{7}$.

12. Find the infinite G. P. whose sum is 4, and whose second term is $\frac{3}{4}$.

13. Find the infinite G. P. whose sum is 9, and whose second term is -4.

14. If every alternate term of a G. P. is taken away, the remaining terms will be in G. P.

15. If all the terms of a G. P. are multiplied by the same number, the products will be in G. P.

16. Show that the reciprocals of the terms of a G. P. are in G. P.

17. By saving 1 cent the first day, 2 cents the second day, 4 cents the third day, and so on, doubling the amount every day, how much would be saved in a month of 30 days?

18. Suppose a body to move eternally as follows: 20 feet the first minute, 19 feet the second minute, $18\frac{1}{20}$ feet the third minute, and so on. Find the limit of the distance passed over.

19. A ball falling from the height of 100 feet rebounds one-fourth the distance, then falling, it rebounds one-fourth the distance, and so on. Find the distance passed through by the ball before it comes to rest.

20. If in problem 31 of exercise 132, $a = 16\frac{1}{12}$, how long will it be before the ball in problem 19 comes to rest?

To fall 100 feet, it takes $\sqrt{100 \div 16\frac{1}{12}}$, or $10\sqrt{\frac{12}{193}}$, seconds; to rebound, or to fall, 25 feet, it takes $\sqrt{25 \div 16\frac{1}{12}}$, or $5\sqrt{\frac{12}{193}}$, seconds; to rebound, or to fall, $6\frac{1}{4}$ feet, it takes $\sqrt{6\frac{1}{4} \div 16\frac{1}{12}}$, or $\frac{5}{2}\sqrt{\frac{12}{193}}$, seconds; and so on.

Hence the time $= 10\sqrt{\frac{12}{193}} + 2(5\sqrt{\frac{12}{193}} + \frac{5}{2}\sqrt{\frac{12}{193}} + \cdots)$

$$= 30\sqrt{\frac{12}{193}} = \frac{60}{193}\sqrt{579} = 7.4805 +.$$

HARMONIC PROGRESSIONS.

415. An **harmonic progression** is a series of numbers whose reciprocals form an A. P.

E.g., the series

$$1,\ \tfrac{1}{3},\ \tfrac{1}{5},\ \tfrac{1}{7},\ \cdots, \text{ or } 4,\ -4,\ -\tfrac{4}{3},\ \cdots$$

is an harmonic progression; for the reciprocals of their terms

$$1,\ 3,\ 5,\ 7,\ \cdots, \text{ or } \tfrac{1}{4},\ -\tfrac{1}{4},\ -\tfrac{3}{4},\ \cdots$$

are in A. P.

416. When three numbers are in harmonic progression (H. P.), the middle term is called the **harmonic mean** of the other two.

417. Let H be the harmonic mean of a and b; then by § 415,

$$\frac{1}{a}, \frac{1}{H}, \frac{1}{b} \text{ are in A. P.}$$

$$\therefore \frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H}.$$

$$\therefore \frac{2}{H} = \frac{1}{a} + \frac{1}{b}, \text{ or } H = \frac{2ab}{a+b}.$$

418. If A and G denote respectively the arithmetic and the geometric mean of a and b, then (§§ 403, 409)

$$A = \frac{a+b}{2},\ G = \sqrt{ab},\ H = \frac{2ab}{a+b}.$$

$$\therefore\ A \times H = \frac{a+b}{2} \times \frac{2ab}{a+b} = ab = G^2.$$

Hence $$A : G = G : H.$$

That is, *the geometric mean of any two numbers is also the geometric mean of their arithmetic and harmonic means.*

419. Problems in H. P. are generally solved by inverting the terms, and making use of the properties of the resulting A. P.

Ex. The fifteenth term of an H. P. is 1/25, and the twenty-third term is 1/41. Find the series.

Let a be the first term, and d the common difference of the corresponding A. P.; then

$$25 = \text{the fifteenth term} = a + 14d,$$

and $$41 = \text{the twenty-third term} = a + 22d.$$

$$\therefore\ d = 2,\ a = -3.$$

Hence the A. P. is $-3,\ -1,\ 1,\ 3,\ 5,\ \cdots,$

and the H. P. is $-\frac{1}{3},\ -1,\ 1,\ \frac{1}{3},\ \frac{1}{5},\ \cdots.$

Exercise 135.

1. Find the sixth term of the series $4,\ 2,\ 1\frac{1}{3},\ \cdots.$

2. Find the eighth term of the series $1\frac{1}{3},\ 1\frac{1}{17},\ 2\frac{2}{13},\ \cdots.$

Find the series in which

3. The second term is 2, and the thirty-first term is $\frac{4}{31}$.

4. The thirty-ninth term is $\frac{1}{11}$, and the fifty-fourth term is $\frac{1}{26}$.

Find the harmonic mean between

5. 2 and 4. **6.** 1 and 13. **7.** $\frac{1}{4}$ and $\frac{1}{10}$.

8. Insert 2 harmonic means between 4 and 12.

9. Insert 3 harmonic means between $2\frac{2}{5}$ and 12.

10. Insert 4 harmonic means between 1 and 6.

11. If a, b, c are in harmonic progression, prove that $a - b : b - c = a : c$.

CHAPTER XXX

PERMUTATIONS AND COMBINATIONS

420. Fundamental principle. *If one thing can be done in* m *ways, and (after it has been done in any one of these ways) a second thing can be done in* n *ways; then the two things can be done in* $m \times n$ *ways.*

Ex. 1. If there are 11 steamers plying between New York and Havana, in how many ways can a man go from New York to Havana and return by a different steamer?

He can make the first passage in 11 ways, with each of which he has the choice of 10 ways of returning; hence he can make the two journeys in 11×10, or 110, ways.

Ex. 2. In how many ways can 3 prizes be given to a class of 10 boys, without giving more than one to the same boy?

The first prize can be given in 10 ways, with each of which the second prize can be given in 9 ways; hence the first two prizes can be given in 10×9 ways. With each of these ways of giving the first two prizes, the third prize can be given in 8 ways; hence the three prizes can be given in $10 \times 9 \times 8$, or 720, ways.

Proof. After the first thing has been done in *any one* of the m ways, the second thing can be done in n different ways; hence there are n ways of doing the two things for *each* of the m ways of doing the first; therefore in all there are mn ways of doing the two things.

This principle is readily extended to the case in which there are three or more things, each of which can be done in a given number of ways.

421. The different ways in which r things can be taken from n things, the *order* of selection or arrangement being

considered, are called the **permutations** *of the* n *things taken* r *at a time.*

Thus, two permutations will be different unless they contain the same things arranged in the same order.

E.g., of the four letters a, b, c, d, taken one at a time, we have the four permutations

$$a, \quad b, \quad c, \quad d.$$

Of these four letters taken two at a time, we have the twelve permutations

$$ab, \; ac, \; ad, \; ba, \; bc, \; bd, \; ca, \; cb, \; cd, \; da, \; db, \; dc.$$

If after each of these permutations we place in turn each of the letters which it does not contain, we shall obtain 24 permutations of the four letters taken three at a time.

The number of permutations of n different things taken r at a time is denoted by the symbol ${}^nP_r$. Thus ${}^9P_2$, ${}^9P_3$, ${}^9P_4$ denote respectively the numbers of permutations of 9 things taken 2, 3, 4 at a time.

422. *To find the number of permutations of* n *dissimilar things taken* r *at a time.*

The number required is the same as the number of ways of filling r places with n different things.

The first place can be filled by any one of the n things, and after this has been filled in any one of these n ways, the second place can be filled in $(n-1)$ ways; hence with n things two places can be filled in $n(n-1)$ ways; that is,

$${}^nP_2 \equiv n(n-1). \tag{1}$$

After the first two places have been filled in any one of these $n(n-1)$ ways, the third place can be filled in $(n-2)$ ways; hence three places can be filled in $n(n-1)(n-2)$ ways; that is,

$${}^nP_3 \equiv n(n-1)(n-2). \tag{2}$$

For like reason, we have

$$^nP_4 \equiv n(n-1)(n-2)(n-3); \tag{3}$$

and so on.

From (1), (2), (3), $\cdots$, we see that in ^nP_r there are ***r** factors,* of which the rth is $n-r+1$; hence

$$^nP_r \equiv n(n-1)(n-2)\cdots(n-r+1). \tag{A}$$

If all the n things are to be taken at a time, $r=n$, and (A) becomes

$$^nP_n \equiv n(n-1)(n-2)\cdots 3\cdot 2\cdot 1. \tag{B}$$

423. The continued product $n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$ is denoted by the symbol $\underline{|n}$, or $n!$, either of which is read 'factorial n.'

Thus $\quad \underline{|4} = 4\cdot 3\cdot 2\cdot 1$; $\quad \underline{|9} = 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot \underline{|4}$.

With this notation (B) in § 422 can be written

$$^nP_n \equiv \underline{|n}. \tag{B}$$

That is, *the number of permutations of **n** different things taken all at a time is factorial **n**.*

Ex. 1. In how many different ways can 7 boys stand in a row?

The number $= {}^7P_7 = 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 5040.$ by (B)

Ex. 2. How many different numbers can be formed with the figures 1, 2, 3, 4, 5, 6, taken four at a time?

The number required $= {}^6P_4 = 6\cdot 5\cdot 4\cdot 3 = 360.$ by (A)

424. *If N denote the number of permutations of **n** things taken all at a time, of which **r** things are alike, **s** others alike, and **t** others alike; then*

$$N = \frac{\underline{|n}}{\underline{|r}\,\underline{|s}\,\underline{|t}}.$$

Proof. Suppose that in any one of the N permutations the r like things were replaced by r dissimilar things; then, from this single permutation, without changing in it the

position of any one of the other $n - r$ things, we could form $\lfloor r$ new permutations. Hence from the N original permutations we could obtain $N\lfloor r$ permutations, in each of which s things would be alike and t others alike.

Similarly, if the s like things were replaced by s dissimilar things, the number of permutations would be $N\lfloor r \lfloor s$, each having t things alike.

Finally, if the t like things were replaced by t dissimilar things, we should obtain $N\lfloor r \lfloor s \lfloor t$ permutations, in which all the things would be dissimilar.

But the number of permutations of n dissimilar things taken all at a time is $\lfloor n$.

Hence $$N\lfloor r \lfloor s \lfloor t = \lfloor n.$$

Therefore $$N = \frac{\lfloor n}{\lfloor r \lfloor s \lfloor t}.$$

Ex. 1. How many different numbers can be formed by the figures 2, 2, 3, 4, 4, 4, 5, 5, 5, 5?

The number $= \dfrac{\lfloor 10}{\lfloor 2 \lfloor 3 \lfloor 4} = 12600.$

Exercise 136.

1. A cabinet maker has 12 patterns of chairs and 7 patterns of tables. In how many ways can he make a chair and a table? *Ans.* 84.

2. There are 9 candidates for a classical, 8 for a mathematical, and 5 for a natural science scholarship. In how many ways can the scholarships be awarded?

3. In how many ways can 2 prizes be awarded to a class of 10 boys, if both prizes may be given to the same boy?

4. Find the number of permutations of the letters in the word *numbers*. How many of these begin with n and end with s?

5. If no digit occur more than once in the same number, how many different numbers can be represented by the 9 digits, taken 2 at a time? 3 at a time? 4 at a time?

6. How many changes can be rung with 5 bells out of 8? How many with the whole peal?

7. How many changes can be rung with 6 bells, the same bell always being last?

8. In how many ways can 15 books be arranged on a shelf, the places of 2 being fixed?

9. Given ${}^nP_4 = 12 \cdot {}^nP_2$; find n.

10. Given $n : {}^nP_3 :: 1 : 20$; find n.

11. Given ${}^nP_3 : {}^{n+2}P_3 :: 5 : 12$; find n.

12. How many different arrangements can be made of the letters of the word *commencement?*

Of the 12 letters, 2 are c's, 3 are m's, 3 are e's, and 2 are n's;

$$\therefore\ N = \frac{\lfloor\underline{12}}{\lfloor\underline{2}\,\lfloor\underline{3}\,\lfloor\underline{3}\,\lfloor\underline{2}} = 3326400.$$

13. Find the number of permutations of the letters of the words *mammalia, caravansera, Mississippi.*

14. In how many ways can 17 balls be arranged, if 7 of them are black, 6 red, and 4 white?

Prove each of the following relations:

15. $n(n-1)(n-2)\cdots(n-r+1)\lfloor\underline{n-r} \equiv \lfloor\underline{n}$.

16. $9 \cdot 8 \cdot 7 \cdot 6/\lfloor\underline{3} = \lfloor\underline{9}/(\lfloor\underline{3}\,\lfloor\underline{5})$.

17. $n(n-1)(n-2)\cdots(n-r+1)/\lfloor\underline{r} \equiv \lfloor\underline{n}/(\lfloor\underline{r}\,\lfloor\underline{n-r})$.

18. $\lfloor\underline{5}\,\lfloor\underline{5}\,(6/5) = \lfloor\underline{6}\,\lfloor\underline{4}$; $\therefore\ \lfloor\underline{5}\,\lfloor\underline{5} < \lfloor\underline{6}\,\lfloor\underline{4}$.

19. $\lfloor\underline{a}\,\lfloor\underline{a}\,(a+1) \div a \equiv \lfloor\underline{a+1}\,\lfloor\underline{a-1}$; $\therefore\ \lfloor\underline{a}\,\lfloor\underline{a} < \lfloor\underline{a+1}\,\lfloor\underline{a-1}$.

20. $\lfloor\underline{a}\,\lfloor\underline{a} < \lfloor\underline{a+1}\,\lfloor\underline{a-1} < \lfloor\underline{a+2}\,\lfloor\underline{a-2} < \lfloor\underline{a+3}\,\lfloor\underline{a-3} < \cdots$.

21. $\lfloor\underline{18-x}\,\lfloor\underline{x}$ is least when $x = 9$.

425. The different ways in which r things can be taken from n things, without regard to the order of selection or arrangement, are called the **combinations** *of the **n** things taken **r** at a time.*

Thus, two combinations will be different unless they both contain precisely the same things.

E.g., of the four letters a, b, c, d, taken two at a time, there are the six combinations

$$ab,\ ac,\ ad,\ bc,\ bd,\ cd.$$

Taken three at a time, there are the four combinations

$$abc,\ abd,\ acd,\ bcd.$$

Taken four at a time, there is one combination only.

The number of combinations of n things taken r at a time is denoted by the symbol ${}^nC_r$.

426. *To find the number of combinations of **n** different things taken **r** at a time.*

Every combination of r different things has $\underline{|r}$ permutations; hence, ${}^nC_r\underline{|r}$ will denote ${}^nP_r$; that is,

$$\begin{aligned}{}^nC_r\underline{|r} &\equiv {}^nP_r\\ &\equiv n(n-1)(n-2)\cdots(n-r+1).\end{aligned}$$

Hence $${}^nC_r \equiv \frac{n(n-1)(n-2)\cdots(n-r+1)}{\underline{|r}}. \qquad \text{(C)}$$

In applying this formula, it is useful to note that the suffix r in the symbol ${}^nC_r$ denotes the number of the factors in both the numerator and denominator of the formula.

Ex. How many groups of 4 boys are there in a class of 17?

$$\text{The number} = {}^{17}C_4 = \frac{17\cdot 16\cdot 15\cdot 14}{4\cdot 3\cdot 2\cdot 1} = 2380.$$

427. In (C) of § 426, multiplying the numerator and denominator of the fraction by $\lfloor\underline{n-r}$, we obtain

$$^nC_r \equiv \frac{n(n-1)(n-2)\cdots(n-r+1)\lfloor\underline{n-r}}{\lfloor\underline{r}\,\lfloor\underline{n-r}},$$

or
$$^nC_r \equiv \frac{\lfloor\underline{n}}{\lfloor\underline{r}\,\lfloor\underline{n-r}}. \tag{D}$$

Substituting $n-r$ for r in (D), we obtain

$$^nC_{n-r} \equiv \frac{\lfloor\underline{n}}{\lfloor\underline{n-r}\,\lfloor\underline{r}}. \tag{1}$$

From (D) and (1), $\quad ^nC_r \equiv {}^nC_{n-r}$. $\qquad$ (E)

The relation in (E) follows also from the consideration that for each group of r things which is selected, there is left a corresponding group of $n-r$ things.

The relation in (E) often enables us to abridge computation.

E.g., $\qquad {}^{15}C_{13} = {}^{15}C_2 = \dfrac{15 \times 14}{2} = 105.$

428. Value of r which renders ^nC_r greatest.

^nC_r, or $\lfloor\underline{n}/(\lfloor\underline{r}\,\lfloor\underline{n-r})$, is greatest when $\lfloor\underline{r}\,\lfloor\underline{n-r}$ is least.

$$\lfloor\underline{a}\,\lfloor\underline{a}\,(a+1) \div a \equiv \lfloor\underline{a+1}\,\lfloor\underline{a-1}, \text{ etc.};$$

$$\therefore\ \lfloor\underline{a}\,\lfloor\underline{a} < \lfloor\underline{a+1}\,\lfloor\underline{a-1} < \lfloor\underline{a+2}\,\lfloor\underline{a-2} < \cdots.$$

Hence, *when n is even*, $\lfloor\underline{r}\,\lfloor\underline{n-r}$ is least, and therefore *^nC_r is greatest*, when $r = n - r$, or $r = n/2$.

Again $\qquad \lfloor\underline{b}\,\lfloor\underline{b+1} \equiv \lfloor\underline{b+1}\,\lfloor\underline{b},$

and $\qquad \lfloor\underline{b+1}\,\lfloor\underline{b} < \lfloor\underline{b+2}\,\lfloor\underline{b-1} < \lfloor\underline{b+3}\,\lfloor\underline{b-2} < \cdots$

Hence *when n is odd*, $\lfloor\underline{r}\,\lfloor\underline{n-r}$ is least, and therefore *^nC_r is greatest*, when $r = n - r \pm 1$, or $r = (n \pm 1)/2$.

E.g., $\lfloor\underline{r}\,\lfloor\underline{18-r}$ is least and $^{18}C_r$ is greatest when $r = 9$.

Again $\lfloor\underline{r}\,\lfloor\underline{15-r}$ is least and $^{15}C_r$ is greatest when $r = 7$ or 8.

Exercise 137.

1. How many combinations can be made of 9 things taken 4 at a time? taken 6 at a time? taken 7 at a time?

The last number $= {}^9C_7 = {}^9C_2 = 36$.

2. How many combinations can be made of 11 things taken 4 at a time? taken 7 at a time?

3. Out of 10 persons 4 are to be chosen by lot. In how many ways can this be done? In all the ways, how often would any one person be chosen?

4. From 14 books in how many ways can a selection of 5 be made, when one specified book is always included? when one specified book is always excluded?

5. On how many days might a person having 15 friends invite a different party of 10? of 12?

6. Given ${}^{\frac{n}{3}}C_2 = 15$, to find n.

7. Given ${}^{n+1}C_4 = 9 \times {}^nC_2$, to find n.

8. In a certain district there are 4 representatives to be elected, and there are 7 candidates. How many different tickets can be made up?

9. Of 8 chemical elements that will unite one with another, how many ternary compounds can be formed? How many binary?

10. There are 15 points in a plane, no 3 of which lie in the same straight line. Find how many straight lines there are, each containing 2 of the points.

11. In a town council there are 25 councillors and 10 aldermen; how many committees can be formed, each consisting of 5 councillors and 3 aldermen?

12. Find the sum of the products of the numbers 1, 3, 5, 2, taken 2 at a time; taken 3 at a time.

13. Find the number of combinations of 55 things taken 50 at a time.

14. If ${}^{2n}C_3 : {}^{n}C_2 = 44 : 3$; find n.

15. If ${}^{n}C_{12} = {}^{n}C_8$; find n; find ${}^{n}C_{17}$; find ${}^{22}C_n$.

16. In a library there are 20 Latin and 6 Greek books; in how many ways can a group of 5 consisting of 3 Latin and 2 Greek books be placed on a shelf?

17. From 3 capitals, 5 other consonants, and 4 other vowels, how many permutations can be made, each beginning with a capital and containing in addition 3 consonants and 2 vowels?

18. If ${}^{18}C_r = {}^{18}C_{r+2}$; find r; find ${}^{r}C_5$.

19. From 7 Englishmen and 4 Americans a committee of 6 is to be formed; in how many ways can this be done when the committee contains, (1) exactly 2 Americans, (2) at least 2 Americans?

20. Of 7 consonants and 4 vowels, how many permutations can be made, each containing 3 consonants and 2 vowels?

21. When repetitions are allowed, ${}^{n}P_r = n^r$, and ${}^{n}P_n = n^n$.

When repetitions are allowed after the first place has been filled in any one of n ways, the second place can be filled in n ways; hence ${}^{n}P_2 = n^2$, etc.

22. In how many ways can 4 prizes be awarded to 10 boys, each boy being eligible for all the prizes?

23. There are 25 points in space, no 4 of which lie in the same plane. Find how many planes there are, each containing 3 of the points.

24. For what value of r is $|\underline{r}\,|\underline{18-r}$ least? $|\underline{r}\,|\underline{21-r}$? $|\underline{r}\,|\underline{45-r}$?

25. For what value of r is ${}^{10}C_r$ greatest? ${}^{11}C_r$? ${}^{15}C_r$? ${}^{20}C_r$? ${}^{31}C_r$?

CHAPTER XXXI

BINOMIAL THEOREM

429. In § 126 the laws of exponents and coefficients of the binomial theorem were proved for positive integral exponents up to 7. These laws hold for all exponents, integral or fractional, positive or negative.

In this chapter we shall prove these laws for any positive integral exponent, and apply them to all exponents.

430. From the *distributive* law for multiplication, it follows that if we take one term from each of any number of binomials and multiply these terms together, we shall obtain a term of the continued product of these binomials; and if we do this in every possible way, we shall obtain all the terms of the continued product of these binomials.

E.g., if we take a letter from each of the three binomials,

$$(a+b)(a+b)(a+b),$$

and multiply the three letters together, we shall obtain a term of the continued product; and if we do this in every possible way, we shall obtain all the terms of this product.

We can take the a's from the three binomials, and we can do this in *one*, and only *one*, way; hence a^3 is a term of the product.

We can take the b from one binomial and the a's from the other two, and we can do this in *three* ways; for the b can be taken from any one of the three binomials; hence $3\,a^2b$ is a term of the product.

We can take the b's from two binomials and a from the third, and we can do this in *three* ways; hence $3\,ab^2$ is a term of the product.

Finally, we can take the b's from the three binomials in *one*, and only *one*, way; hence b^3 is a term of the product.

Hence $\quad (a+b)(a+b)(a+b) \equiv a^3 + 3\,a^2b + 3\,ab^2 + b^3.$

431. Binomial theorem. Suppose we have

$$(a+b)(a+b)(a+b)\cdots \text{ to } n \text{ factors.} \tag{1}$$

If we take a letter from each of the n binomials, and multiply these letters together, we shall obtain a term of the continued product; and if we do this in every possible way, we shall obtain all the terms of this product.

We can take the a's from all the binomials in *one*, and only *one*, way; hence a^n is one term of the product.

We can take b from one binomial and the a's from the remaining $(n-1)$ binomials, and we can do this in as many ways as one b can be taken from the n binomials, *i.e.*, n, or ${}^nC_1$ ways; hence ${}^nC_1 \cdot a^{n-1}b$ is a term of the product.

Again, we can take the b's from two binomials, and the a's from the remaining $(n-2)$ binomials, and we can do this in as many ways as two b's can be taken from the n binomials, *i.e.*, ${}^nC_2$ ways; hence ${}^nC_2 \cdot a^{n-2}b^2$ is a term of the product.

And, in general, we can take the b's from r binomials (where r is any positive integer not greater than n), and the a's from the remaining $(n-r)$ binomials, and we can do this in as many ways as r b's can be taken from the n binomials, *i.e.*, ${}^nC_r$ ways; hence ${}^nC_r \cdot a^{n-r}b^r$ is the $(r+1)$th, or general, term of the product.

The b's can be taken from the n binomials in *one*, and only *one*, way; hence we have the term b^n, and this is what the general term ${}^nC_r a^{n-r}b^r$ becomes when $r=n$.

Hence $\quad (a+b)(a+b)(a+b)\cdots$ to n factors

$$\equiv a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \cdots + {}^nC_ra^{n-r}b^r + \cdots + b^n. \tag{2}$$

If we substitute for ${}^nC_1$, ${}^nC_2$, etc., their values as given in § 426, we obtain (n denoting any positive integer)

$$(a+b)^n \equiv a^n + na^{n-1}b + \frac{n(n-1)}{\underline{|2}}a^{n-2}b^2 + \cdots$$

$$+ \frac{n(n-1)(n-2)\cdots(n-r+1)}{\underline{|r}}a^{n-r}b^r + \cdots + b^n. \tag{3}$$

Identity (2) or (3) is the symbolic statement of the **binomial theorem.**

The second member of either is called the **expansion** of $(a+b)^n$.

Observe that (3) states in symbols the laws in § 126, and that therefore (3) can be written out by these laws.

Note that the sum of the exponents of a and b in any term is n.

Ex. Expand $(x^{-2}-y^3)^4$.

$$\begin{aligned}(x^{-2}-y^3)^4 &\equiv [(x^{-2})+(-y^3)]^4\\ &\equiv (x^{-2})^4+4(x^{-2})^3(-y^3)+6(x^{-2})^2(-y^3)^2+4(x^{-2})(-y^3)^3\\ &\qquad +(-y^3)^4\\ &\equiv x^{-8}-4x^{-6}y^3+6x^{-4}y^6-4x^{-2}y^9+y^{12}.\end{aligned}$$

432. In the expansion of $(a+b)^n$, the *general term*

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{\underline{|r}}a^{n-r}b^r = \textbf{the } (\boldsymbol{r}+\mathbf{1})\textbf{th term.}$$

Observe that there are r factors in both the numerator and denominator of the coefficient of the $(r+1)$th term.

By giving to r the proper value, we can find any term in the expansion of $(a+b)^n$.

When n is a positive integer, the coefficient of the $(r+1)$th term becomes zero for any value of r greater than n; hence *there are* $\boldsymbol{n+1}$ *terms in the expansion of* $(\boldsymbol{a}+\boldsymbol{b})^n$.

Thus, when $r=n+1$,

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{\underline{|r}}=\frac{n(n-1)(n-2)\cdots(n-n)}{\underline{|r}}\equiv 0.$$

Ex. Find the seventh term of the expansion of $(4x/5-5/2x)^9$.

Here $a=4x/5,\ b=-5/(2x),\ n=9,\ r=6.$

Substituting these values in the formula, we have

$$\begin{aligned}\text{the seventh term} &= \frac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\left(\frac{4x}{5}\right)^3\left(\frac{-5}{2x}\right)^6\\ &= 10500\,x^{-3}.\end{aligned}$$

433. The coefficients of the expansion in (2) of § 431 are

$$1,\ {}^nC_1,\ {}^nC_2,\ {}^nC_3,\ \cdots,\ {}^nC_{n-2},\ {}^nC_{n-1},\ {}^nC_n;$$

hence the $(r+1)$th term from the beginning is ${}^nC_r a^{n-r}b^r$, and the $(r+1)$th term from the end is ${}^nC_{n-r}a^r b^{n-r}$.

But ${}^nC_r = {}^nC_{n-r}$ for all values of r (§ 427).

Hence, in the expansion of $(a+b)^n$, *the coefficients of any two terms equidistant from the beginning and the end are the same, so that the coefficients of the last half of the expansion can be written from those of the first half.*

434. If, in identity (3) of § 431, we put $a=1$ and $b=x$, we have

$$(1+x)^n \equiv 1 + nx + \frac{n(n-1)}{\underline{|2}}x^2 + \cdots + \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}x^r + \cdots + x^n. \tag{1}$$

This is a convenient form of the *binomial theorem*, and one which is often used.

Observe that this form includes all cases; *e.g.*, if we want to find $(a+b)^n$, we have

$$(a+b)^n \equiv \left\{a\left(1+\frac{b}{a}\right)\right\}^n \equiv a^n\left(1+\frac{b}{a}\right)^n \equiv a^n\left(1+n\frac{b}{a}+\cdots\right)$$
$$\equiv a^n + na^{n-1}b + \cdots.$$

435. In (1) of § 434 the coefficients of $x, x^2, x^3, \cdots, x^n$ are the values of ${}^nC_1, {}^nC_2, {}^nC_3, \cdots, {}^nC_n$; hence (1) can be written

$$(1+x)^n = 1 + {}^nC_1x + {}^nC_2x^2 + \cdots + {}^nC_rx^r + \cdots + {}^nC_nx^n. \tag{1}$$

Putting $x=1$, we obtain

$$2^n = 1 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_r + \cdots + {}^nC_n. \tag{2}$$

That is, *the sum of the coefficients in the expansion of $(1+x)^n$, or $(a+b)^n$, is 2^n.*

From (2) it follows also that *the sum of all the combinations that can be made of n things, taken 1, 2, $\cdots$, n at a time, is 2^n-1.*

Exercise 138.

By the laws in § 126 write the expansion of:

1. $(3x^2 - 2y)^4$.
2. $(2a^2 - 3b)^5$.
3. $(c^2 + b^2)^6$.
4. $(3x^2 + y)^5$.
5. $(2 - 3x^2)^6$.
6. $(r^{-2} - b^{\frac{1}{2}})^4$.
7. $(r^2 - 3n^{\frac{1}{2}})^5$.
8. $(2x/3 - 3/x)^6$.
9. $(x^{-\frac{1}{3}} - a^{\frac{2}{3}})^4$.
10. $(a^{-2} - x^{-3})^5$.
11. $(x^{-\frac{3}{5}} - 2c^{\frac{1}{3}})^4$.
12. $(1 - 1/x)^6$.
13. $(xy - a^{-\frac{2}{3}})^5$.
14. $(2x/3 - a/c)^6$.
15. $(x^{-\frac{2}{3}} - 2y^{-\frac{3}{4}})^5$.
16. Find the 3d term in the expansion of $(a - 3b)^{10}$.
17. Find the 7th term in the expansion of $(1 - x)^{10}$.
18. Find the middle term in the expansion of $(1 + x)^{10}$.
19. Find the middle term in the expansion of $(2x - 3y)^8$.
20. Find the 18th term in the expansion of $(1 + x)^{20}$.
21. Find the 7th term in the expansion of
$$[4x/5 - 5/(2x)]^9.$$
22. Find the 17th term in the expansion of $(x^2 - 1/x)^{20}$.

436. Binomial theorem, exponent fractional or negative.

When the exponent of a binomial is fractional or negative, the laws in § 126, or, what is the same thing, the formula

$$(a+b)^n \equiv a^n + na^{n-1}b + \frac{n(n-1)}{\underline{|2}}a^{n-2}b^2 + \cdots$$
$$+ \frac{n(n-1)(n-2)\cdots(n-r+1)}{\underline{|r}}a^{n-r}b^r + \cdots, \quad (1)$$

gives an infinite series; for in this case no one of the factors n, $n-1$, $n-2$, etc., in the $(r+1)$th term can ever be zero.

When, however, $\boldsymbol{r}$ *increases without limit, the sum of* $\boldsymbol{r}$ *terms of this series will approach* $(\boldsymbol{a}+\boldsymbol{b})^n$ *as its limit, provided the first term of the binomial is arithmetically greater than the*

second term. That is, when n is fractional or negative, the infinite series in (1) is the expansion of $(a+b)^n$ provided $a > b$ arithmetically.

A proof of this theorem is too difficult to be given here. For a rigorous proof, see Taylor's "Calculus," § 98.

Ex. 1. Expand $(c^{-1} - d^2)^{-\frac{3}{5}}$ and find the general term.

Applying the laws in § 126 we obtain

$$
\begin{aligned}
(c^{-1} - d^2)^{-\frac{3}{5}} &\equiv [(c^{-1}) + (-d^2)]^{-\frac{3}{5}} \\
&\equiv (c^{-1})^{-\frac{3}{5}} - \tfrac{3}{5}(c^{-1})^{-\frac{8}{5}}(-d^2) + \tfrac{12}{25}(c^{-1})^{-\frac{13}{5}}(-d^2)^2 \\
&\qquad - \tfrac{52}{125}(c^{-1})^{-\frac{18}{5}}(-d^2)^3 + \cdots \qquad (1) \\
&\equiv c^{\frac{3}{5}} + \tfrac{3}{5}c^{\frac{8}{5}}d^2 + \tfrac{12}{25}c^{\frac{13}{5}}d^4 + \tfrac{52}{125}c^{\frac{18}{5}}d^6 + \cdots. \qquad (2)
\end{aligned}
$$

The two distinct steps, that of applying the laws to obtain (1) and that of performing the indicated operations in (1) to obtain (2), must be taken separately.

In performing the operations indicated in (1), first note the number of negative numeral factors in a term to determine the quality of its numeral coefficient. Thus in the fourth term there are four negative factors, $-\frac{52}{125}$ and $(-1)^3$.

Substituting in the general term for n, a, and b their values $-\frac{3}{5}$, c^{-1}, and $-d^2$, we obtain

$$
\text{the } (r+1)\text{th term} = \frac{(-\frac{3}{5})(-\frac{8}{5})(-\frac{13}{5})\cdots(-\frac{3}{5}-r+1)}{\lfloor\!\underline{r}}(c^{-1})^{-\frac{3}{5}-r}(-d^2)^r
$$

$$
\equiv \frac{3\cdot 8\cdot 13\cdots(5r-2)}{5^r\lfloor\!\underline{r}}\,c^{\frac{3}{5}+r}d^{2r}. \qquad (3)
$$

Since there are r factors in the numerator in (3), the term involves the $2r$th power of -1, which is $+1$.

In (2), by this article d^2 must be arithmetically less than c^{-1}.

Ex. 2. Expand $1/(1+x)$ and find the general term.

Applying the laws in § 126, we obtain

$$
\begin{aligned}
(1+x)^{-1} &\equiv 1^{-1} - 1\cdot 1^{-2}\cdot x + 1\cdot 1^{-3}x^2 - 1\cdot 1^{-4}x^3 + \cdots \\
&\equiv 1 - x + x^2 - x^3 + x^4 - \cdots. \qquad (1)
\end{aligned}
$$

$$
\text{The } (r+1)\text{th term} = \frac{(-1)(-2)\cdots(-r)}{\lfloor\!\underline{r}}1^{-1-r}x^r \equiv (-1)^r x^r. \qquad (2)
$$

In (1), by this article x is limited to values between -1 and $+1$.

Ex. 3. $(1+x)^{-2} \equiv 1^{-2} - 2 \cdot 1^{-3} \cdot x + 3 \cdot 1^{-4} \cdot x^2 - 4 \cdot 1^{-5} x^3 + \cdots$

$$\equiv 1 - 2x + 3x^2 - 4x^3 + \cdots.$$

The $(r+1)$th term $\equiv \dfrac{(-2)(-3)\cdots(-r-1)}{\lfloor r} 1^{-2-r} x^r \equiv (-1)^r (r+1) x^r.$

Ex. 4. Expand $1/\sqrt{1-x}$, or $(1-x)^{-\frac{1}{2}}$.

$$(1-x)^{-\frac{1}{2}} \equiv 1^{-\frac{1}{2}} - \tfrac{1}{2} \cdot 1^{-\frac{3}{2}}(-x) + \tfrac{3}{8} \cdot 1^{-\frac{5}{2}}(-x)^2 - \tfrac{5}{16} \cdot 1^{-\frac{7}{2}}(-x)^3 + \cdots$$

$$\equiv 1 + \tfrac{1}{2}x + \tfrac{3}{8}x^2 + \tfrac{5}{16}x^3 + \cdots.$$

The $(r+1)$th term $\equiv \dfrac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{1}{2}-r+1)}{\lfloor r} 1^{-\frac{1}{2}-r}(-x)^r$

$$\equiv \frac{1 \cdot 3 \cdot 5 \cdots (2r-1)}{2^r \lfloor r} x^r.$$

Ex. 5. Find the cube root of 127.

$$127 = 125 + 2 = 5^3 + 2.$$

$$\therefore \sqrt[3]{127} = (5^3+2)^{\frac{1}{3}}.$$

$$= (5^3)^{\frac{1}{3}} + \tfrac{1}{3}(5^3)^{-\frac{2}{3}} 2 - \tfrac{1}{9}(5^3)^{-\frac{5}{3}} 2^2 + \tfrac{5}{81}(5^3)^{-\frac{8}{3}} 2^3 + \cdots$$

$$= 5 + \frac{1}{3} \cdot \frac{2}{5^2} - \frac{1}{9} \cdot \frac{4}{5^5} + \frac{5}{81} \cdot \frac{8}{5^8} + \cdots$$

$$= 5 + 0.0266666 - 0.0001422 + 0.0000012 - \cdots$$

$$= 5.0265255 -.$$

The smaller the ratio of the second term of the binomial to the first, the more rapidly the successive terms of the expansion decrease, and therefore the fewer the terms it is necessary to find.

Here we put $127 = 125 + 2$, because 125 is the perfect cube which makes the ratio of the second term to the first the smallest.

Exercise 139.

Expand to four terms:

1. $(1-x)^{-1}$.	5. $(1+x)^{-3}$.	9. $(1-5x)^{\frac{3}{5}}$.
2. $(1-x)^{-2}$.	6. $(1+2x)^{-4}$.	10. $(b^{\frac{1}{2}} - c^{-\frac{3}{2}})^{-\frac{3}{4}}$.
3. $(1-x)^{-3}$.	7. $(2-x)^{-3}$.	11. $a/\sqrt{x^2-y^3}$.
4. $(1-x)^{-4}$.	8. $(1-3x)^{-\frac{2}{3}}$.	12. $b/(a^{\frac{2}{3}} - b^{-\frac{1}{2}})$.

Find the general term in the expansion of:

13. $(1-x)^{-5}$. **15.** $(1-x)^{-\frac{3}{2}}$. **17.** $(1-2x)^{-\frac{5}{2}}$.

14. $(1-x)^{-n}$. **16.** $(1+x)^{-\frac{2}{3}}$. **18.** $(1+3x)^{-\frac{4}{3}}$.

In its expansion find the:

19. Sixth term and eleventh term of $(\frac{1}{2}a - b\sqrt{b})^{22}$.

20. Fifth term of $(1-a^2)^{-\frac{3}{2}}$.

21. Seventh term of $(x^{-1} - y^{\frac{1}{2}})^{\frac{4}{3}}$.

22. Third term and eleventh term of $(1+2x)^{\frac{11}{2}}$.

23. Fifth term of $(c^{-2} + e^{-\frac{1}{2}})^{-4}$.

24. Sixth term of $(x^{-\frac{2}{3}} - a^2b^{\frac{3}{4}})^{-\frac{3}{5}}$.

Find to four places of decimals the value of:

25. $\sqrt[3]{31}$. **27.** $\sqrt[5]{29}$. **29.** $\sqrt[4]{620}$. **31.** $\sqrt[3]{998}$.

26. $\sqrt[4]{17}$. **28.** $\sqrt[3]{122}$. **30.** $\sqrt[5]{31}$. **32.** $\sqrt[5]{3128}$.

Expand to four terms:

33. $(8+12a)^{\frac{2}{3}}$.

34. $(1-3x)^{\frac{1}{3}}$.

35. $(1-3x)^{-\frac{1}{3}}$.

36. $(a^2 + c^{\frac{1}{2}})^{\frac{3}{4}}$.

37. $(c-d^2)^{\frac{3}{5}}$.

38. $(9+2x)^{\frac{1}{2}}$.

39. $(4a-8x)^{-\frac{1}{2}}$.

40. $(c^2a^{-\frac{1}{2}} - b^2e^{-\frac{2}{3}})^{-\frac{3}{5}}$.

41. $\dfrac{a}{(cb^{-\frac{2}{3}} - x^2y^{-\frac{1}{2}})^{\frac{3}{4}}}$.

42. Find the general term in each of the examples from 33 to 39 inclusive.

CHAPTER XXXII

LOGARITHMS

437. The *exponent* which the base a must have in order to equal the number N is called the **logarithm** *of N to the base a.*

That is, if
$$a^x = N, \tag{1}$$

x is the logarithm of N to the base a, which is written

$$x = \log_a N. \tag{2}$$

Equations (1) and (2) are equivalent; (2) is the *logarithmic* form of writing the relation between a, x, and N, given in (1).

E.g., since $3^2 = 9$, 2 is the logarithm of 9 to the base 3, or $2 = \log_3 9$.

Since $2^4 = 16$, 4 is the logarithm of 16 to the base 2, or $4 = \log_2 16$.

Since $2^{-3} = 1/8$, $-3 = \log_2 (1/8)$.

Since $4^{\frac{3}{2}} = 8$, $3/2 = \log_4 8$.

Review §§ 52, 336, 338, 339, 386 on exponents.

Exercise 140.

1. Express each of the following relations in the logarithmic form:

$$2^3 = 8,\ 3^4 = 81,\ 4^3 = 64,\ 12^2 = 144,\ 6^3 = 216,\ n^c = b.$$

2. Express each of the following relations in the exponential form:

$$\log_5 125 = 3,\ \log_2 32 = 5,\ \log_4 64 = 3,\ \log_3 81 = 4,\ \log_c M = b.$$

3. When the base is 3, what are the logarithms of 1, 3, 9, 27, 81, 243, 729?

4. When the base is 4, what are the logarithms of 1, 4, 16, 64, 256, 1024?

5. When the base is 2, what are the logarithms of 1, 1/2, 1/4, 1/8, 1/16, 1/32, 1/64, 1/128, 1/256?

6. When the base is 10, what are the logarithms of 1, 10, 100, 1000, 10000, 100000, 0.1, 0.01, 0.001, 0.0001, 0.00001?

7. When the base is 3, and the logarithms are 0, 1, 2, 3, 4, -1, -2, -3, -4, what are the numbers?

438. The logarithms of all *arithmetic* numbers to any given base constitute a **system of logarithms.**

Since $1^x \equiv 1$, 1 cannot be the *base* of a system of logarithms. Any *arithmetic* number except 1 can evidently be taken as the base of a system of logarithms.

Since logarithms are exponents, from the general properties of exponents, we obtain the general

PROPERTIES OF LOGARITHMS TO ANY BASE.

439. *The logarithm of* 1 *is zero.*

Proof. $a^0 = 1, \quad \therefore \log_a 1 = 0.$

440. *The logarithm of the base itself is* 1.

Proof. $a^1 = a, \quad \therefore \log_a a = 1.$

441. *The logarithm of a* **product** *is equal to the* **sum** *of the logarithms of its factors.*

Proof. Let $M = a^x,\ N = a^y;$

then $M \times N = a^{x+y}.$ § 345

Hence $\log_a(M \times N) = x + y = \log_a M + \log_a N.$

E.g., $\log_4(16 \times 64) = \log_4 16 + \log_4 64 = 2 + 3 = 5.$

442. *The logarithm of a* **quotient** *is equal to the logarithm of the dividend* **minus** *the logarithm of the divisor.*

Proof. Let $M = a^x,\ N = a^y$;

then $M \div N = a^{x-y}$. § 346

Hence $\log_a (M \div N) = x - y = \log_a M - \log_a N$.

E.g., $\log_3 (243 \div 27) = \log_3 243 - \log_3 27$

$= 5 - 3 = 2$.

443. *The logarithm of any* **power** *of a number is equal to the logarithm of the number* **multiplied** *by the exponent of the power.*

Proof. Let $M = a^x$;

then, for all real values of p, we have

$$M^p = a^{px}. \qquad \text{§ 348}$$

Hence $\log_a (M^p) = px = p \log_a M$.

E.g., $\log_4 (16^3) = 3 \cdot \log_4 16 = 3 \times 2 = 6$;

$\log_3 (81^{\frac{3}{4}}) = \frac{3}{4} \log_3 81 = \frac{3}{4} \times 4 = 3$;

and $\log_5 (25^{-\frac{3}{2}}) = -\frac{3}{2} \log_5 25 = -\frac{3}{2} \times 2 = -3$.

444. By § 443, the logarithm of any positive integral power of a number is equal to the logarithm of the number multiplied by the exponent of the power; and the logarithm of any root of a number is equal to the logarithm of the number divided by the index of the root.

Ex. 1. Given

$\log_{10} 2 = .30103$ and $\log_{10} 3 = .47712$; find $\log_{10} \sqrt[3]{720}$.

$\log_{10} \sqrt[3]{720} = \frac{1}{3} \log_{10} (2^3 \times 3^2 \times 10)$ § 443

$= \frac{1}{3}(3 \log_{10} 2 + 2 \log_{10} 3 + \log_{10} 10)$ §§ 441, 443

$= \frac{1}{3}(.90309 + .95424 + 1) = .95244$.

Ex. 2. $\log_a[\sqrt[3]{x^2} \div (b^3c^{\frac{5}{4}})] \equiv \log_a x^{\frac{2}{3}} - \log_a(b^3c^{\frac{5}{4}})$ § 442

$\equiv \log_a x^{\frac{2}{3}} - (\log_a b^3 + \log_a c^{\frac{5}{4}})$ § 441

$\equiv \frac{2}{3}\log_a x - 3\log_a b - \frac{5}{4}\log_a c.$ § 443

445. *If a series of numbers are in geometric progression, their logarithms are in arithmetic progression.*

E.g., if $N = 1/27,\ 1/9,\ 1/3,\ 1,\ 3,\ 9,\ 27, \cdots$

$\log_3 N = -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3, \cdots.$

Proof. The logarithms of the terms of the G. P.

$$N, \quad Nr, \quad \cdots, \quad Nr^n,$$

are $\log_a N, \quad \log_a N + \log_a r, \quad \cdots, \quad \log_a N + n\log_a r,$

which is an A. P. whose common difference is $\log_a r$.

Exercise 141.

Express $\log_a y$ in terms of $\log_a b$, $\log_a c$, $\log_a x$, and $\log_a z$, having given the following equations:

1. $y = x^3b^{\frac{2}{3}}c^5.$
2. $y = \sqrt[3]{z^2}\cdot\sqrt{c^6}.$
3. $y = \dfrac{b^3x^2z}{c^{\frac{2}{3}}}.$
4. $y = \sqrt[7]{b^2x^4z^3}.$
5. $y = \sqrt{xz^5}\cdot\sqrt{b^2c^7}.$
6. $y = \dfrac{\sqrt{x^3zb^5}}{\sqrt[3]{x^2z^4c^5}}.$

Given $\log_{10} 2 = .3010$, $\log_{10} 3 = .4771$, find:

7. $\log_{10} 4$; $\log_{10} 5$; $\log_{10} 6$; $\log_{10} 8$; $\log_{10} 9$; $\log_{10} 10$.

   $\log_{10} 5 = \log_{10} 10 - \log_{10} 2 = 1 - .3010 = .6990.$

8. $\log_{10} 12.$
9. $\log_{10} 16.$
10. $\log_{10} 30.$
11. $\log_{10} 50.$
12. $\log_{10}(3/2).$
13. $\log_{10}(6/5).$
14. $\log_{10}\sqrt{600}.$
15. $\log_{10}\sqrt[3]{120}.$

16. Between what integral numbers does $\log_{10} N$ lie, when N lies between 10 and 100? Between 1 and 10? Between .1 and 1? Between .01 and .1? Between .001 and .01?

446. *If the base of logarithms is greater than* 1,

(i) *The logarithm of a number is positive or negative, according as the number is greater or less than* 1.

(ii) *The logarithm of an infinite is infinite; and the logarithm of an infinitesimal is a negative infinite, or, as it is often stated, the logarithm of zero is negative infinity.*

Proof. By § 437 x is the logarithm of a^x to the base a.

Let $a > 1$; then, by the principles of exponents, we know that

$$\text{if } a^x > 1,\ x > 0;\ \text{if } a^x < 1,\ x < 0;\ \text{hence (i).}$$

If $\quad a^x = \infty,\ x = \infty;$ if $a^x \doteq 0,\ x = -\infty;$ hence (ii).

COMMON LOGARITHMS.

447. The logarithms used for *abridging arithmetic computations* are those to the base 10; for this reason logarithms to the base 10 are called **common logarithms.**

Thus the common logarithm of a number answers the question, '*What power of* 10 *is the number?*'

Most numbers are incommensurable powers of 10; hence most common logarithms are incommensurable numbers, whose approximate values we express decimally.

Hereafter in this chapter when no base is written, the base 10 is to be understood.

When a logarithm is negative, for convenience it is expressed as a *negative* integer plus a *positive decimal.*

E.g., the common logarithm of any number

between 10 and 100 is $^+1$ + a positive fraction;

between 1 and 10 is 0 + a positive fraction;

between 0.1 and 1 is $^-1$ + a positive fraction;

between 0.01 and 0.1 is $^-2$ + a positive fraction.

448. The *integral* part of a logarithm is called the **characteristic,** and the *positive decimal* part the **mantissa.**

A negative characteristic is usually written in the form

$$\bar{1}, \text{ or } 9-10;\ \bar{2}, \text{ or } 8-10;\ \bar{3}, \text{ or } 7-10; \text{ etc.}$$

E.g., $\log 434.1 = 2.63759$; $^{+}2$ is the characteristic and $.^{+}63759$ is the mantissa: $\log 0.0769 = \bar{2}.88593$, or $8.88593 - 10$; $\bar{2}$, or $8 - 10$, is the negative characteristic, and $.^{+}88593$ is the mantissa. The sign $^{-}$ is written over the 2 to show that it affects the characteristic alone.

449. The *characteristic* of the common logarithm of any number is found by the following simple rule:

When the number is greater than 1, *the characteristic is positive and arithmetically one less than the number of digits to the left of the decimal point; when the number is less than* 1, *the characteristic is negative and arithmetically one greater than the number of zeros between the decimal point and the first significant figure.*

E.g., 785 lies between 10^2 and 10^3;

hence $$\log 785 = 2 + \text{a mantissa}.$$

Again 0.0078 lies between 10^{-3} and 10^{-2};

hence $$\log 0.0078 = -3 + \text{a mantissa}.$$

Proof. Let N denote a number which has m digits to the left of the decimal point; then N lies between 10^{m-1} and 10^m;

that is, $$N = 10^{(m-1) + \text{a fraction}}.$$

$$\therefore \log N = (m-1) + \text{a mantissa}.$$

Again let N denote a decimal which has m zeros between the decimal point and the first significant figure; then N lies between $10^{-(m+1)}$ and 10^{-m};

that is, $$N = 10^{-(m+1) + \text{a fraction}}.$$

$$\therefore \log N = -(m+1) + \text{a mantissa}.$$

450. *The common logarithms of numbers which differ only in the position of the decimal point have the same mantissa.*

Proof. When a change is made in the position of the decimal point, the number is multiplied or divided by some integral power of 10; that is, an integer is added to, or subtracted from, the logarithm, and therefore *its mantissa is not changed.*

E.g.,
$$\log 1054.3 = 3.02296,$$
$$\log 1.0543 = 0.02296,$$
$$\log .010543 = 8.02296 - 10, \text{ or } \bar{2}.02296.$$

451. When a negative logarithm is to be divided by a number, and its negative characteristic is not exactly divisible by that number, the logarithm must be so modified in form that the negative integral part will be exactly divisible by the number.

Ex. Given $\log 0.0785 = \bar{2}.8949$; find $\log \sqrt[7]{0.0785}$.

$$\text{Log } \sqrt[7]{0.0785} = \tfrac{1}{7} \log 0.0785 = \tfrac{1}{7}(\bar{2}.8949)$$
$$= \tfrac{1}{7}(\bar{7}. + 5.8949) = \bar{1}.8421.$$

Adding $-5+5$ to the logarithm does not change its value and makes its negative part divisible by 7.

Exercise 142.

1. Log $427.32 = 2.6307$. Find $\log 42732$, $\log 42.732$.

2. Log $23.95 = 1.3793$. Find $\log 23950$, $\log 239.5$, $\log 239500$, $\log 0.002395$, $\log 0.0002395$, $\log 2395$.

3. Log $4398 = 3.64326$. Find $\log \sqrt{0.4398}$, $\log \sqrt[3]{0.4398}$, $\log \sqrt[4]{439.8}$, $\log \sqrt[6]{0.04398}$, $\log \sqrt[7]{0.004398}$.

4. Log $674.8 = 2.82918$. Find $\log \sqrt[3]{0.6748}$, $\log \sqrt[5]{0.6748}$, $\log \sqrt[6]{0.06748}$, $\log \sqrt[7]{0.06748}$, $\log \sqrt[8]{0.006748}$.

452. Tables of logarithms. Common logarithms have two great practical advantages: (i) Characteristics are known

by § 449, so that only mantissas are tabulated; (ii) mantissas are determined by the sequence of digits (§ 450), so that the mantissas of integral numbers only are tabulated.

At the close of this chapter will be found a table which contains the *mantissas* of the common logarithms of all numbers from 1 to 999 correct to four decimal places.

NOTE. Tables are published which give the logarithms of all numbers from 1 to 99999 calculated to seven places of decimals; these are called 'seven-place' logarithms. For many purposes, however, the four-place or five-place logarithms are sufficiently accurate.

From a table of logarithms we can obtain:

(i) The logarithm of a given number;

(ii) The number corresponding to a given logarithm.

453. *To find the logarithm of a given number.*

Ex. 1. Find log 7.85.

By § 450, the required mantissa is the mantissa of log 785.

Look in column headed "N" for 78. Passing along this line to the column headed 5, we find .8949, the required mantissa.

Prefixing the characteristic, we have

$$\log 7.85 = 0.8949.$$

Ex. 2. Find log 4273.2.

When the number contains more than three significant figures, we must use the principle that when the difference of two numbers is *small* compared with either of them, the difference of the numbers is *approximately* proportional to the difference of their logarithms.

By § 450, the required mantissa is that of log 427.32.

$$\text{The mantissa of } \log 427 = .6304.$$

$$\text{The mantissa of } \log 428 = .6314.$$

That is, an increase of 1 in the number causes an increase of .0010 in the mantissa; hence an increase of .32 in the number will cause an increase of .32 of *approximately* .0010, or .0003, in the mantissa.

Adding .0003 to the mantissa of log 427, and prefixing the characteristic, we have

$$\log 4273.2 = 3.6307.$$

Ex. 3. Find log 0.0006049.

By § 450, the required mantissa is that of log 604.9.

$$\text{The mantissa of } \log 604 = .7810.$$

Also, an increase of 1 in the number causes an increase of .0008 in the mantissa; hence .9 of .0008, or .0007, must be added to .7810.

$$\therefore \log 0.0006049 = \bar{4}.7817, \text{ or } 6.7817 - 10.$$

To find log 30 or log 3, find mantissa of log 300.

Exercise 143.

Find, from the table, the logarithm of the numbers:

1. 8.	**5.** 703.	**9.** 0.05307.	**13.** 7.4803.
2. 50.	**6.** 7.89.	**10.** 78542.	**14.** 2063.4.
3. 6.3.	**7.** 0.178.	**11.** 0.50438.	**15.** 0.0087741.
4. 374.	**8.** 3.476.	**12.** 0.00716.	**16.** 0.017423.

454. *To find a number when its logarithm is given.*

Ex. 1. Find the number of which the logarithm is 3.8954.

Look in the table for the mantissa .8954. It is found in line 78 and in column 6; hence

$$.8954 = \text{the mantissa of } \log 786.$$

$$\therefore 3.8954 = \log 7860;$$

or 7860 is the number whose logarithm is 3.8954.

Ex. 2. Find the number of which the logarithm is 1.6290.

Look in the table for the mantissa .6290. It cannot be found; but the next less mantissa is .6284, and the next greater is .6294.

Also, $\quad .6284 = \text{mantissa of } \log 425,$

and $\quad .6294 = \text{mantissa of } \log 426.$

That is, an increase of .0010 in the mantissa causes an increase of 1 in the number; hence an increase of .0006 in the mantissa will cause an increase of *approximately* $\frac{6}{10}$ of 1, or .6, in the number; hence

$$.6290 = \text{the mantissa of } \log 425.6;$$

$$\therefore 1.6290 = \log 42.56.$$

Ex. 3. Find the number of which the logarithm is $\bar{3}.8418$.

Look in the table for the mantissa .8418. It cannot be found; but the next less mantissa is .8414, and

$$.8414 = \text{mantissa of } \log 694.$$

Also, an increase of .0006 in the mantissa causes an increase of 1 in the number; hence an increase of .0004 in the mantissa will cause an increase of $\frac{4}{6}$ of 1, or .66 in the number; hence

$$.8418 = \text{the mantissa of } \log 694.66.$$

$$\therefore\ \bar{3}.8418 = \log 0.0069466.$$

Exercise 144.

Find the number of which the logarithm is:

1. 1.8797.	6. 8.1648 – 10.	11. 3.7425.
2. 7.6284 – 10.	7. 9.3178 – 10.	12. 7.1342 – 10.
3. 0.2165.	8. 1.6482.	13. 3.7045.
4. 2.7364.	9. 8.5209 – 10.	14. 8.7982 – 10.
5. 4.0095.	10. 3.8016.	15. 3.4793.

455. The **cologarithm** of a number is the logarithm of its reciprocal.

That is, $\quad \text{colog } N = \log(1 \div N) = -\log N.$

To make the *fractional* part of the cologarithm *positive*, if $\log N > 0$ and < 10, colog N is written

$$(10 - \log N) - 10;$$

if $\log N > 10$ and < 20, colog N is written

$$(20 - \log N) - 20.$$

E.g., $\quad \text{colog } 0.0574 = -(\bar{2}.7589) = 1.2411;$

$$\text{colog } 432 = (10 - 2.6263) - 10 = 7.3737 - 10;$$

$$\text{colog } 345000000000 = (20 - 11.5378) - 20 = 8.4622 - 20.$$

Instead of subtracting the logarithm of a divisor, we can, by § 87, add its cologarithm.

Ex. 1. Find the value of $\dfrac{15.08 \times 0.0723}{0.0534 \times 7.238}$.

$$\begin{aligned} \log\ 15.08 &= 1.1784 \\ \log 0.0723 &= 8.8591 - 10 \\ \text{colog}\ 0.0534 &= 1.2725 \\ \text{colog}\ 7.238 &= \underline{9.1404 - 10} \end{aligned}$$

Add, $\log(\text{fraction}) = 0.4504 \qquad = \log 2.8213.$

Hence the fraction $= 2.8213$.

Ex. 2. Find the value of $0.0543 \times 6.34 \times (-5.178)$.

$$\begin{aligned} \log 0.0543 &= 8.7348 - 10 \\ \log\ 6.34 &= 0.8021 \\ \log\ 5.178 &= \underline{0.7141} \end{aligned}$$

Add, $\log(\text{product}) = 0.2510 \qquad = \log 1.7824.$

Hence the product is -1.7824.

By logarithms we obtain simply the *arithmetic* value of the result; its *quality* must be determined by the laws of quality.

Ex. 3. Find the value of $\sqrt[5]{\dfrac{5.42 \times 427.^2}{3.24^4 \times 0.0231^{\frac{1}{2}}}}$.

$$\begin{aligned} \log 5.42 &= 0.7340 & &= 0.7340 \\ 2 \log 427. &= (2.6304) \times 2 & &= 5.2608 \\ 4\ \text{colog}\ 3.24 &= (9.4895 - 10) \times 4 & &= 7.9580 - 10 \\ \tfrac{1}{2}\ \text{colog}\ 0.0231 &= (1.6364) \div 2 & &= \underline{0.8182} \\ & & & 5)\overline{4.7710} \\ & & \therefore \log(\text{root}) &= 0.9542 \\ & & \therefore \text{root} &= 9.00 \end{aligned}$$

456. An **exponential equation** is one in which the unknown appears in an exponent; as $2^x = 5$, $x^x = 10$.

Such equations are solved by the aid of logarithms.

Ex. 1. Solve $\qquad 3^{2x} - 14 \times 3^x + 45 = 0. \qquad (1)$

Factor (1), $\qquad (3^x - 9)(3^x - 5) = 0. \qquad (2)$

Equation (2) is equivalent to the two equations

$$3^x = 9, \qquad (3) \qquad\qquad 3^x = 5. \qquad (4)$$

From (3), $\qquad x = 2.$

From (4), $\qquad x \log 3 = \log 5.$

$$\therefore\ x = \frac{\log 5}{\log 3} = \frac{0.6990}{0.4771} = 1.4649.$$

Hence the roots of (1) are 2 and 1.4649.

Exercise 145.

Find by logarithms the value of:

1. $742.8 \times 0.02374.$
2. $0.3527 \times 0.00572.$
3. $78.42 \times 0.000437.$
4. $5234 \times (-0.03671).$
5. $3.246 \times (-0.0746).$
6. $-4.278 \times (-0.357).$
7. $4743 \div 327.4.$
8. $9.345 \div (-0.0765).$
9. $\dfrac{2.476 \times (-0.742)}{73.81 \times (-0.00121)}.$
10. $\dfrac{321 \times (-48.1) \times (357)}{421 \times (-741) \times (4.21)}.$
11. $5^{\frac{2}{3}}.$
12. $0.021^{\frac{5}{2}}.$
13. $0.532^{8}.$
14. $(\frac{17}{54})^{9}.$
15. $714.2^{\frac{2}{5}}.$
16. $(\frac{471}{891})^{7}.$
17. $(\frac{15}{23})^{2.7}.$
18. $(3\frac{2}{5})^{1.27}.$
19. $4.71^{3.205}.$
20. $\sqrt{\dfrac{0.035^3 \times 54.2 \times 785^{\frac{1}{4}} \times 0.0742}{4.72^{\frac{1}{3}} \times 7.14^{\frac{1}{5}} \times 8.47^{\frac{1}{6}}}}.$
21. $\sqrt[3]{\dfrac{0.0427^2 \times 5.27 \times 0.875^4}{7.421^{\frac{1}{4}} \times \sqrt{1.74} \times \sqrt{0.00215}}}.$
22. $\sqrt[5]{\dfrac{0.714^{\frac{1}{2}} \times 0.1371^{\frac{1}{3}} \times 0.0718^{\frac{1}{4}}}{0.524^2 \times 0.742^{\frac{1}{5}} \times 0.0527^{\frac{1}{2}}}}.$

Solve each of the following equations:

23. $31^x = 23$.
25. $5^x = 800$.
27. $5^{x-3} = 8^{2x+1}$.
24. $0.3^x = 0.8$.
26. $12^x = 3528$.
28. $a^{2x}b^{3x} = c^5$.
29. $2^{3x}5^{2x-1} = 4^{5x}3^{x+1}$.
30. $4^{2x} - 15(4^x) + 56 = 0$.

COMPOUND INTEREST AND ANNUITIES.

457. *To find the compound interest, \$ I, and amount, \$ M, of a given principal, \$ P, in* n *years, \$* r *being the interest on \$1 for 1 year.*

Let \$ $R =$ the amount of \$1 in 1 year; then $R = 1 + r$, and the amount of \$ P at the end of the first year is \$ PR; and since this is the principal for the second year, the amount at the end of the second year is \$ $PR \times R$, or \$ PR^2. For like reason the amount at the end of the third year is \$ PR^3, and so on; hence the amount in n years is \$ PR^n; that is,

$$M = PR^n, \text{ or } P(1+r)^n. \tag{1}$$

Hence

$$I = P(R^n - 1). \tag{2}$$

If the interest is payable semi-annually, the amount of \$ P in 1/2 a year will be \$ $P(1 + r/2)$; hence, as n years equals $2n$ half-years,

$$M = P(1 + r/2)^{2n}. \tag{3}$$

Similarly, if the interest is payable quarterly,

$$M = P(1 + r/4)^{4n}. \tag{4}$$

Ex. Find the time in which a sum of money will double itself at ten per cent compound interest, interest to be "converted into principal" semi-annually.

Here $1 + r/2 = 1.05$. Let $P = 1$; then $M = 2$.

Substituting these values in (3), we obtain

$$2 = (1.05)^{2n}.$$

$$\therefore \log 2 = 2n \cdot \log 1.05.$$

$$\therefore n = \frac{\log 2}{2 \log 1.05} = \frac{0.3010}{0.0424} = 7.1 \text{ years. } Ans.$$

458. Present value and discount. Let $\$P$ denote the present value of the sum $\$M$ due in n years, at the rate r; then evidently in n years at the rate r, $\$P$ will amount to $\$M$; hence

$$M = PR^n, \text{ or } P = MR^{-n}.$$

Let $\$D$ be the discount; then

$$D = M - P = M(1 - R^{-n}).$$

459. An **annuity** is a fixed sum of money that is payable once a year, or at more frequent regular intervals, under certain stated conditions. An *Annuity Certain* is one payable for a fixed number of years. A *Life Annuity* is one payable during the lifetime of a person. A *Perpetual Annuity*, or *Perpetuity*, is one that is to continue forever, as, for instance, the rent of a freehold estate.

460. *To find the amount of an annuity left unpaid for a given number of years, allowing compound interest.*

Let $\$A$ be the annuity, n the number of years, $\$R$ the amount of one dollar in one year, $\$M$ the required amount. Then evidently the number of dollars due at the end of the

$$\begin{aligned}
&\text{First year} &&= A; \\
&\text{Second year} &&= AR + A; \\
&\text{Third year} &&= AR^2 + AR + A; \\
&n\text{th year} &&= AR^{n-1} + AR^{n-2} + \cdots + AR + A \\
& &&= \frac{A(R^n - 1)}{R - 1}.
\end{aligned}$$

That is,
$$M = \frac{A}{r}(R^n - 1). \qquad (1)$$

Ex. 1. Find the amount of an annuity of $100 in 20 years, allowing compound interest at $4\frac{1}{2}$ per cent.

$$M = \frac{A}{r}(R^n - 1) = \frac{100(1.045^{20} - 1)}{0.045}.$$

By logarithms, $1.045^{20} = 2.4117$.

$$\therefore\ M = \frac{141.17}{0.045} = 3137.11.$$

Hence the amount of the annuity is \$3137.11.

Ex. 2. What sum must be set aside annually that it may amount to \$50,000 in 10 years at 6 per cent compound interest?

From (1), $A = \dfrac{Mr}{R^n - 1} = \dfrac{50{,}000 \times 0.06}{1.06^{10} - 1} = 3793.37.$

Hence the required sum is \$3793.37.

461. *To find the present value of an annuity of \$A payable at the end of each of n successive years.*

Let \$$P$ denote the present value; then the amount of \$$P$ in n years will equal the amount of the annuity in the same time: that is,

$$PR^n = A(R^n - 1)r^{-1}. \tag{1}$$

$$\therefore\ P = A(1 - R^{-n})r^{-1}. \tag{2}$$

If the annuity is *perpetual*, then $n = \infty$, $R^{-n} \doteq 0$, and (2) becomes

$$P = Ar^{-1}.$$

Exercise 146.

1. Write out the logarithmic equations for finding each of the four numbers, M, R, P, n.

2. In what time, at 5 per cent compound interest, will \$100 amount to \$1000?

3. Find the time in which a sum will double itself at 4 per cent compound interest.

4. Find in how many years \$1000 will become \$2500 at 10 per cent compound interest.

5. Find the present value of \$10,000 due 8 years hence at 5 per cent compound interest.

6. Find the amount of \$1 at 5 per cent compound interest in a century.

7. Show that money will increase more than thirteen-thousand-fold in a century at 10 per cent compound interest.

8. If A leaves B \$1000 a year to accumulate for 3 years at 4 per cent compound interest, find what amount B should receive.

9. Find the present value of the legacy in example 8.

10. Find the present value, at 5 per cent, of an estate of \$1000 a year to be entered on immediately.

11. A freehold estate worth \$120 a year is sold for \$4000; find the rate of interest.

12. A man has a capital of \$20,000, for which he receives interest at 5 per cent; if he spends \$1800 every year, show that he will be ruined before the end of the 17th year.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8388	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

CHAPTER XXXIII

GRAPHIC SOLUTION OF EQUATIONS AND SYSTEMS

462. Let XX' and YY' be any two fixed straight lines at right angles to each other at O. Let the directions OX and OY be *positive* directions; then the directions OX' and OY' will be *negative* directions.

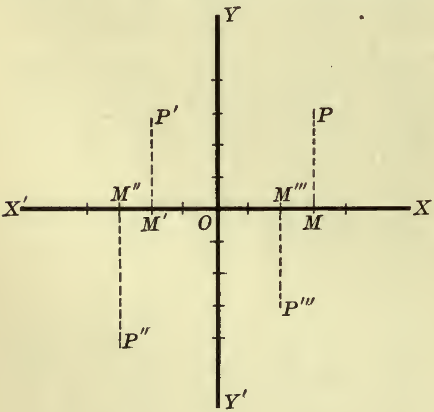


FIG. 1

The lines XX' and YY' are called **axes of reference,** and their intersection O, the **origin.**

From P, any point in the plane of the axes, draw PM parallel to YY'; then the position of P will be determined when we know both the *lengths* and the *directions* of the lines OM and MP.

The line OM, or its numerical measure, is called the **abscissa** of the point P; and MP, or its numerical measure, is called the **ordinate** of P. The abscissa and ordinate together are called the **coördinates** of P.

E.g., OM' and $M'P'$ are the coördinates of P'; the abscissa, OM', is negative, and the ordinate, $M'P'$, is positive. OM''', the abscissa of P''', is positive, and $M'''P'''$, its ordinate, is negative.

An abscissa is usually denoted by the letter x, and an ordinate by y.

Observe that the *numerical measure* of OM or MP is a *positive* number, if it extends in the direction OX or OY; and a *negative* number if it extends in the direction OX' or OY'.

The axis XX' is called the **axis of abscissas**, or the **x-axis**; and YY', the **axis of ordinates**, or the **y-axis**.

The point whose coördinates are x and y is denoted by (x, y).

E.g., $(2, -3)$ denotes the point of which the abscissa is 2, and the ordinate -3.

We use a system of coördinates analogous to that explained above whenever we locate a city by giving its latitude and longitude; the equator is one axis, and the assumed meridian the other.

Ex. Plot the point $(-2, 3)$; $(-3, -4)$.

In the figure lay off $OM' = -2$, and on $M'P'$ parallel to YY' lay off $M'P' = +3$; then P' is the point $(-2, 3)$.

To plot $(-3, -4)$, lay off $OM'' = -3$, and on $M''P''$ parallel to YY' lay off $M''P'' = -4$; then P'' is the point $(-3, -4)$.

The lines XX' and YY' divide the plane into four equal parts called **quadrants**, which are numbered as follows: XOY is the *first quadrant*, YOX' the *second*, $X'OY'$ the *third*, and $Y'OX$ the *fourth*.

Exercise 147.

1. Plot the point $(2, 3)$; $(4, 7)$; $(3, -5)$; $(-2, +3)$; $(-3, +5)$; $(4, -2)$; $(-2, -3)$; $(-5, -3)$; $(-2, 4)$; $(-4, -1)$; $(0, 0)$.

2. In which quadrant is $(^+a, ^+b)$? $(^+a, ^-b)$? $(^-a, ^+b)$? $(^-a, ^-b)$?

3. What is the quality of x and of y, when the point (x, y) is in the first quadrant? Second quadrant? Third quadrant? Fourth quadrant?

4. In which quadrants can the point (x, y) be, when x is positive? x negative? y positive? y negative?

5. In what line is the point $(x, 0)$? $(0, y)$?

6. Where is the point $(0, 0)$? $(4, 0)$? $(-3, 0)$? $(0, 2)$? $(0, -5)$?

463. Graphic solution of equations in x and y.

The **locus**, or *graph*, of an equation in x and y is the line or lines which include all the points, and only those, whose coördinates satisfy the equation.

Ex. 1. Draw the locus of $y = x^2 - x - 6$. (1)

If in (1) we put $x = -3, -2, -1, \cdots$, we obtain

when $\quad x = -3, \ -2, \ -1, \quad 0, \ 1/2, \quad 1, \quad 2, \ 3, \ 4, \ \cdots,$

$\quad y = 6, \quad 0, \ -4, \ -6, \ -6\frac{1}{4}, \ -6, \ -4, \ 0, \ 6, \ \cdots.$

Drawing the axes XX' and YY' in fig. 2, and assuming $O1$ as the linear unit, we plot the points

$$(-3, 6), \ (-2, 0), \ (-1, -4), \ (0, -6), \ \cdots.$$

The relative positions of these points indicate the form of a curve through them.

Whenever there is any doubt about the form of this curve between any two plotted points, as between $(0, -6)$ and $(1, -6)$, one or more intermediate points should be found and plotted.

As x increases indefinitely from 3, y (or $x^2 - x - 6$) continues positive and increases indefinitely; hence the locus has an infinite branch in the first quadrant. As x decreases indefinitely from -2, y continues positive and increases indefinitely; hence the locus has an infinite branch in the second quadrant.

Drawing a smooth curve through the plotted points we obtain the curve ABC in fig. 2, which, with its infinite branches, is the locus of equation (1).

This curve is called the locus of the equation because each and every *real* solution of equation (1) is the coördinates of some point on the curve.

Imaginary or complex solutions of an equation cannot be represented by the coördinates of any points in the plane XOY, since by definition the coördinates of every point in this plane are real.

NOTE. The pupil should use coördinate or cross-section paper, and with a hard pencil draw the loci of equations neatly and accurately.

$y = x^2 - x - 6.$

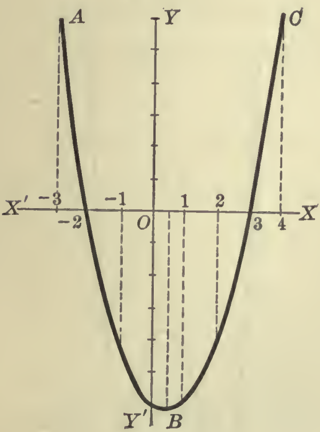


FIG. 2

$y = x^3 - 2x.$

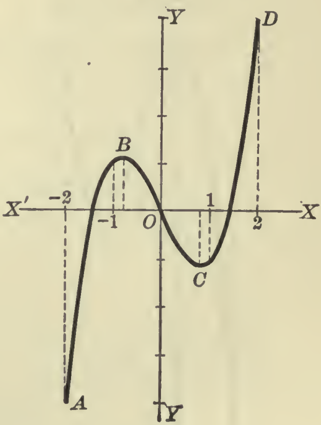


FIG. 3

Ex. 2. Draw the locus of $y = x^3 - 2x$. (1)

When $x = -2,\ -\sqrt{2},\ -1,\ -0.8,\ 0,\ 0.8,\ 1,\ \sqrt{2},\ 2,\ \cdots,$

$y = -4,\ 0,\ 1,\ 1.1,\ 0,\ -1.1,\ -1,\ 0,\ 4,\ \cdots.$

As x increases indefinitely from 2, y (or $x^3 - 2x$) continues positive and increases indefinitely; hence the locus has an infinite branch in the first quadrant.

As x decreases indefinitely from -2, y continues negative and arithmetically increases without limit; hence there is an infinite branch in the third quadrant.

Plotting these points, as in fig. 3, and tracing a smooth curve through them, we obtain the curve $ABCD$, which, with its infinite branches, is the locus of (1).

Ex. 3. Draw the locus of $y = x^3 - 3x^2 + 4$.

When $\quad x = -3/2,\ -1,\ -\frac{1}{2},\ 0,\ 1/2,\ 1,\ 2,\ 3,\ \cdots,$

$\quad y = -6.1,\ \ 0,\ 3.1,\ 4,\ 3.4,\ 2,\ 0,\ 4,\ \cdots.$

As x increases from 2, y increases indefinitely from 0; and, as x decreases from -1, y decreases indefinitely from 0.

The locus is the curve *ABCD* in fig. 4, which has one infinite branch in the first quadrant and another in the third.

$y = x^3 - 3x^2 + 4.$

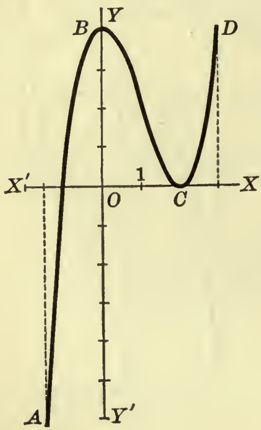


FIG. 4

$y = x^4 + x^3 - 3x^2 - x + 2.$

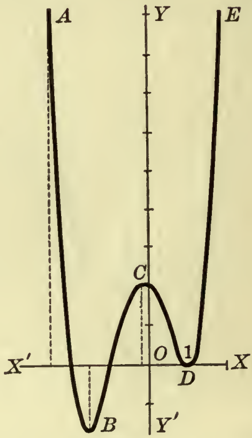


FIG. 5

Ex. 4. Draw the locus of $y = x^4 + x^3 - 3x^2 - x + 2$.

When $\quad x = -\frac{5}{2},\ -2,\ \ -\frac{3}{2},\ -1,\ -\frac{1}{2},\ -.2,\ 0,\ \ \frac{1}{2},\ 1,\ \ \frac{3}{2},\ \cdots,$

$\quad y = 9.2,\ \ 0,\ -1.6,\ \ 0,\ 1.7,\ 2.08,\ 2,\ 0.9,\ 0,\ 2.2,\ \cdots.$

The locus is the curve in fig. 5, which has one infinite branch in the first quadrant and another in the second.

The foregoing examples illustrate how each *real* solution of an equation in x and y is the coördinates of some point in the locus; hence, by the coördinates of its points, the locus of an equation in x and y gives all its infinite number of real solutions.

Exercise 148.

Draw the locus of each of the following equations, stating in which quadrants the infinite branches lie:

1. $y = x^2 - 4x - 5$.

2. $y = x - x^2 + 6$.

3. $y = x^3 - 4x$.

4. $y = x^3 - 3x^2 - 4x + 12$.

5. $y = x^3 - x^2 - 6x$.

6. $y = x^4 - 5x^2 + 4$.

7. Using one set of axes, draw the loci of

$$y = x,\ y = x + 2,\ y = x - 2.$$

Observe that these loci are *parallel straight* lines.

8. Using one set of axes, draw the loci of

$$y = 2x,\ y = 2x + 3,\ y = 2x - 3.$$

9. Draw the locus of

$$y = 3x - 2, \text{ of } y = -2x + 1, \text{ of } 2y = 4x - 6.$$

These examples illustrate the truth that *the locus of any linear equation in x and y is a straight line.*

Hence to draw the locus of any linear equation we can plot two of its points and draw a straight line through them.

10. Draw the locus of

$$y = 4, \text{ of } y = -3, \text{ of } x = 3, \text{ of } x = -4.$$

Observe that each of these loci is parallel to one of the axes.

11. From the origin O as a centre and with a radius 5, draw the circumference of a circle. Draw the ordinate PM of any point P on this circumference and the radius OP. Denote the coördinates of P by x and y. Then from the right-angled triangle OMP, we obtain $x^2 + y^2 = 5^2$.

What then is the locus of $x^2 + y^2 = 25$?

12. Draw the locus of $x^2 + y^2 = 9$, of $x^2 + y^2 = 16$.

Examples 11 and 12 illustrate the truth that *the locus of any equation of the form $x^2 + y^2 = r^2$ is the circumference of a circle whose centre is at the origin and whose radius is r.*

13. Draw the locus of $\quad 4x^2 + 9y^2 = 36. \qquad (1)$

Here $$y = \pm \tfrac{2}{3}\sqrt{9 - x^2}.$$

Evidently -3 is the least value of x which will render y real; hence no part of the locus can lie to the left of the line $x = -3$. For like reason no part of the locus can lie to the right of the line $x = 3$.

When $\quad x = -3,\ -2.5,\ \ -2,\ \ -1,\ \ 0,\ \ 1,\ \ 2,\ 3,$

$\qquad\quad y = 0,\ \pm 1.1,\ \pm 1.5,\ \pm 1.9,\ \pm 2,\ \pm 1.9,\ \pm 1.5,\ 0.$

The locus is the ellipse $RASB$ (fig. 8, page 439), the semi-axes being 3 and 2.

Observe that in (1) the coefficients of x^2 and y^2 are unequal, while in examples 11 and 12 they are equal.

14. Draw the locus of $x^2 + 4y^2 = 4$.

15. Draw the locus of $x^2 - 4y^2 = 4$.

Here $$y = \pm 1/2\sqrt{x^2 - 4}. \qquad (1)$$

When $x > 2$ or < -2, the values of y in (1) are real; when x lies between -2 and $+2$, y is imaginary; hence there is an infinite branch in each of the four quadrants, but no point of the locus lies between the lines $x = -2$ and $x = 2$.

16. Draw the locus of $y^2 = 4x$.

17. Using one set of axes, draw the loci of

$$x - y = -3 \text{ and } x + y = 1.$$

Observe that these equations are independent and consistent, and that the one and only point common to their loci is $(-1, 2)$.

18. Using one set of axes, draw the loci of

$$2x - y = -1 \text{ and } 2x - y = -3.$$

Observe that these equations are inconsistent, and that their loci are parallel and hence have no point in common.

19. Using one set of axes, draw the loci of

$$2x + y = 1 \text{ and } 6x + 3y = 3.$$

Observe that these equations are equivalent, and that their loci coincide and hence have all points in common.

20. What is the greatest number of points in which a straight line can cut the locus in fig. 2? In fig. 3? In fig. 4? In fig. 5? Compare each answer with the degree of the equation of each locus.

464. Graphic solution of systems of equations.

Ex. 1. By the aid of loci discuss the system

$$\left.\begin{aligned} ax + by &= c, & (1)\\ a'x + b'y &= c'. & (2)\end{aligned}\right\} (a)$$

Let the locus of (1) be the straight line MN, and that of (2) the line RP. Then the coördinates of the point P, which is common to both loci, will be the solution common to (1) and (2), or the solution of the system (a). By measuring the coördinates OA and AP, the numerical solution of the system could be obtained.

This example illustrates graphically the theorem in § 207.

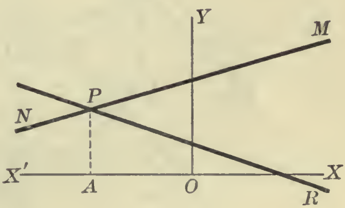


FIG. 6

The loci will have one, and only one, point in common,

if $$a/a' \neq b/b',$$

i.e., if (1) and (2) are independent and consistent (§ 207).

The loci will coincide throughout their whole extent,

if $$a/a' \neq b/b' = c/c',$$

i.e., if (1) and (2) are equivalent (§§ 207, 357).

The loci will be parallel and have no point in common,

if $$a/a' = b/b' \text{ and } a/a' \neq c/c',$$

i.e., if (1) and (2) are inconsistent (§§ 207, 357).

Ex. 2. By the aid of loci, discuss the system

$$\left.\begin{aligned} x^2 + y^2 &= 25, \quad &(1)\\ y &= x + c, \quad &(2)\end{aligned}\right\}(a)$$

for different values of c.

The locus of (1) is the circle $PP'RP''$; and, if $c = 1$, the graph of (2) is the straight line MN; hence the coördinates of the two points P and R are the two solutions of system (a).

By measurement we find the two solutions to be 3, 4 and $-4, -3$.

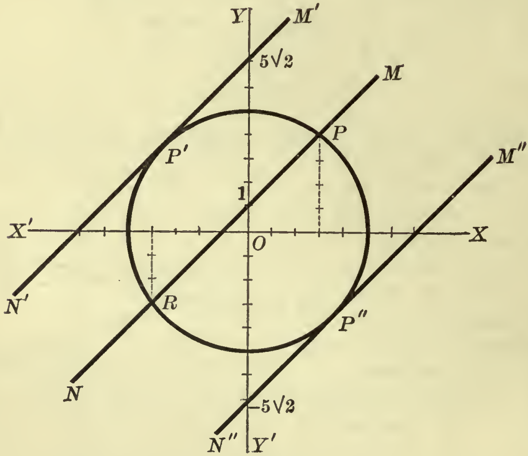


Fig. 7

As c increases, the locus MN moves upward parallel to itself, and P and R approach P'.

When $c = 5\sqrt{2}$, the locus of (2) is the tangent $N'M'$, and the two solutions of the system are equal.

Similarly, when $c = -5\sqrt{2}$, the locus of (2) is $M''N''$.

When $c < 5\sqrt{2}$ and $> -5\sqrt{2}$, the locus of (2) lies between $N'M'$ and $N''M''$, and the two solutions of the system are real and unequal.

When $c > 5\sqrt{2}$ or $< -5\sqrt{2}$, the locus of (2) does not cut the circle, and both solutions of the system are imaginary or complex.

Ex. 3. By aid of loci discuss the system

$$\left.\begin{aligned} 4x^2 + 9y^2 &= 36, && (1)\\ x^2 + y^2 &= r^2 && (2)\end{aligned}\right\} (a)$$

for different values of r.

The locus of (1) is the ellipse $ARBS$, in which $OA=3$ and $OR=2$.

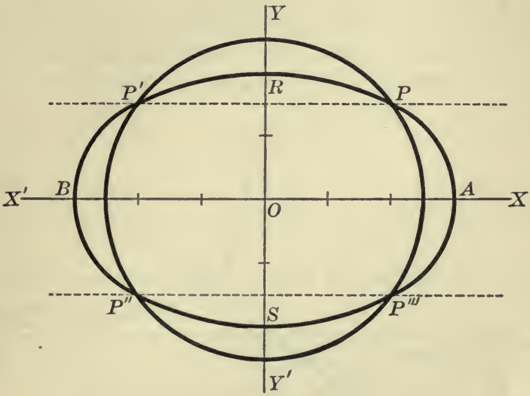


FIG. 8

If $r = 5/2$, the locus of (2) is the circle $PP'P''P'''$, and the four solutions of the system are the coördinates of the four points P, P', P'', P''', and thus are real and unequal.

If $r = 3$, the circle will be tangent to the ellipse at A and B; hence two solutions of the system will be 3, 0, and the other two -3, 0.

If $r = 2$, the circle will be tangent to the ellipse at R and S.

If $r < 2$ or > 3, the two loci will have no common points, and all four solutions of the system will be imaginary or complex.

When $r = 5/2$, by clearing (2) of fractions and then subtracting it from (1) we obtain $5y^2 = 11$, the locus of which is the parallel lines PP' and $P'''P''$. These lines cut either the ellipse or the circle in all the points which are common to these curves, and only in these points. This illustrates the equivalency of system (a) to the system

$$\left.\begin{aligned} 4x^2 + 9y^2 &= 36,\\ 5y^2 &= 11.\end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} x^2 + y^2 &= \tfrac{25}{4},\\ 5y^2 &= 11.\end{aligned}\right\}$$

Ex. 4. By aid of loci discuss the system

$$\left.\begin{aligned} xy &= 12, && (1)\\ y &= mx + n, && (2)\end{aligned}\right\}(a)$$

for different values of m and n.

The locus of (1) is the curves AB and CD, whose infinite branches approach the axes.

When $n = 0$ and $m = 3/4$, the locus of (2) is the line PP', and the two solutions of system (a) are the coördinates of the points P and P'.

Let $m \doteq 0$; then P will move out along the infinite branch PB, and P' along the infinite branch $P'C$; that is, $y \doteq 0$ and $x = +\infty$ or $-\infty$.

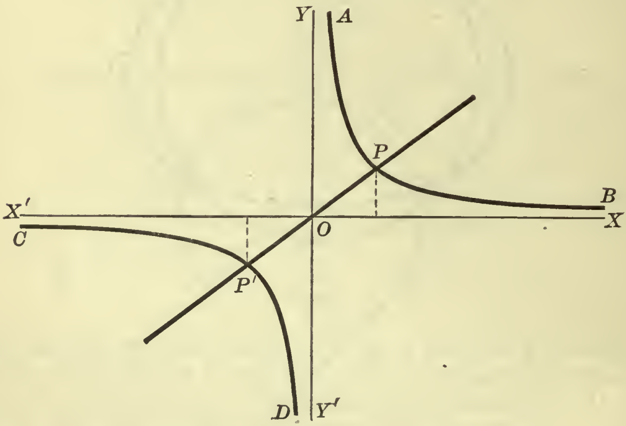


FIG. 9

Again, when $n = 0$, the two solutions of system (a) are

$$2\sqrt{3/m},\ 2\sqrt{3m},\ \text{and}\ -2\sqrt{3/m},\ -2\sqrt{3m}. \qquad (3)$$

For $m = 0$, the solutions in (3) assume the forms

$$a/0,\ 0\ \text{and}\ -a/0,\ 0;$$

hence equation (1) and $y = 0$ are inconsistent, and system (a) is then impossible. This agrees with the figure, for the locus of $y = 0$ coincides with $X'OX$, and does not intersect the locus of (1).

When m is negative, the solutions in (3) become imaginary. This agrees with the figure; for when m is negative, x and y in $y = mx$ are

opposite in quality, and hence the locus of $y = mx$ will lie in the second and fourth quadrants, and will not cut the locus of (1).

If $m = 0$ and $n \neq 0$, the locus of (2) will be parallel to XOX', and will cut the locus of (1) in only one point; hence system (*a*) will be defective in one solution.

Exercise 149.

1. By aid of loci show that system (*a*) is equivalent to the four systems in (*b*).

$$\left.\begin{aligned} x^2 + y^2 = 25 \\ xy = 12 \end{aligned}\right\}. \qquad (a)$$

$$\left.\begin{aligned} x + y = 7 \\ x - 1 = 1 \end{aligned}\right\} \quad \left.\begin{aligned} x + y = 7 \\ x - y = -1 \end{aligned}\right\} \quad \left.\begin{aligned} x + y = -7 \\ x - y = 1 \end{aligned}\right\} \quad \left.\begin{aligned} x + y = -7 \\ x - y = -1 \end{aligned}\right\}. \qquad (b)$$

2. By aid of loci show that the following six systems are equivalent:

$$\left.\begin{aligned} x^2 + y^2 = 25 \\ x^2 - y^2 = 7 \end{aligned}\right\} \qquad \left.\begin{aligned} x^2 + y^2 = 25 \\ x^2 = 16 \end{aligned}\right\} \qquad \left.\begin{aligned} x^2 + y^2 = 25 \\ y^2 = 9 \end{aligned}\right\}$$

$$\left.\begin{aligned} x^2 - y^2 = 7 \\ x^2 = 16 \end{aligned}\right\} \qquad \left.\begin{aligned} x^2 - y^2 = 7 \\ y^2 = 9 \end{aligned}\right\} \qquad \left.\begin{aligned} x^2 = 16 \\ y^2 = 9 \end{aligned}\right\}.$$

GRAPHIC SOLUTION OF EQUATIONS IN X.

465. A variable whose value depends upon one or more other variables is called a **dependent variable**, or a **function** of those variables. A variable which does not depend upon any other variable for its value is called an **independent variable**.

E.g., x^3, $2x^2 - 3x + 7$, or $x^4 - 7x^2 + 9$, is a function of the independent variable x.

Again, y in each of the equations in this chapter is a function of the independent variable x.

The symbol $f(x)$, read 'function x,' is used to denote any function of x.

The symbols $f(a)$, $f(2)$, $f(1)$ represent *the values of* $f(x)$ *when* $x = a$, 2, 1, respectively.

E.g., if $f(x) = x^3 + x$, then

$$f(a) = a^3 + a,\ f(2) = 2^3 + 2 = 10,\ f(1) = 2.$$

Since $f(x)$ denotes any function of x, $y = f(x)$ denotes any equation in x and y, when the equation is solved for y. Thus, any one of the equations in the first ten examples in exercise 146 is a particular case of $y = f(x)$.

466. A **continuous variable** is a variable which in passing from one value to another passes successively through all intermediate values.

A function, as $f(x)$, is said to be continuous between $x = a$ and $x = b$, if when x changes continuously from a to b, $f(x)$ varies continuously from $f(a)$ to $f(b)$. In other words, $f(x)$ is continuous between $x = a$ and $x = b$, when the locus of $y = f(x)$ is an unbroken curve between the lines $x = a$ and $x = b$.

E.g., the time since any past event varies continuously. The velocity acquired by a falling body and the distance fallen are continuous functions of the time *of falling*.

In each of the four examples in § 463, y is a continuous function of x for all real values of x.

In example 2 of § 464, y in equation (1) is real and a continuous function of x between $x = -5$ and $x = +5$.

The examples in § 463 illustrate the fact that

Any rational integral function of x is a continuous function.

In what follows we shall use $f(x)$ to denote a *rational integral* function of x.

467. The ordinates of the points in the locus of $y = x^2 - x - 6$ in fig. 1, of § 463, are the successive values of $x^2 - x - 6$ corresponding to successive values of x; hence, the locus of $y = f(x)$ is often called the *graph of* $f(x)$.

E.g., in fig. 2, while x *increases* continuously from -3 to zero, the function $x^2 - x - 6$ *decreases* continuously from $+6$ though zero to -6; and while x *increases* from zero to $+4$, $x^2 - x - 6$ first decreases from -6 and then *increases* to $+6$.

Again, in fig. 3, while x increases continuously from -2 to -0.8, the function $x^3 - 2x$ *increases* continuously from -4 to $+1.1$; while x increases from -0.8 to $+0.8$, $x^3 - 2x$ *decreases* from $+1.1$ to -1.1; while x increases from $+0.8$ to $+2$, $x^3 - 2x$ *increases* from -1.1 to 4.

In like manner, in the other figures, the pupil should follow the changes in $f(x)$ as x increases.

468. The abscissas of the points in which the graph of $f(x)$ cuts or touches the axis of x are the real values of x for which $f(x)$ is zero; that is, they are the real roots of the equation $f(x) = 0$.

At a point of tangency the graph is properly said to touch the axis of x in *two coincident points.*

E.g., from the graph in fig. 2, we learn that one root of the equation $x^2 - x - 6 = 0$ is -2 and the other is 3.

From the graph in fig. 3, we learn that the three roots of the equation $x^3 - 2x = 0$ are $-\sqrt{2}$, 0, and $\sqrt{2}$.

In fig. 4, the graph cuts the axis of x at $(-1, 0)$ and touches it at $(2, 0)$; hence, one root of $x^3 - 3x^2 + 4 = 0$ is -1 and the other two roots are 2 each.

Hence, to find the real roots of $f(x) = 0$, we can draw the graph of $f(x)$, or the locus of $y = f(x)$, and measure the abscissas of the points of intersection and tangency with the x-axis.

Exercise 150.

Construct the graph of $f(x)$, and find approximately the real roots of each of the following equations:

1. $x^2 + x - 2 = 0$.
2. $x^3 + 2x - 5 = 0$.
3. $x^3 - 3x + 4 = 0$.
4. $x^3 - 3x^2 - 4x + 11 = 0$.
5. $x^3 - 4x^2 - 6x - 8 = 0$.
6. $x^4 - 4x^3 - 3x + 2 = 0$.

CHAPTER XXXIV

THEORY OF EQUATIONS

469. Horner's method of synthetic division.

Let it be required to divide

$$Ax^3 + Bx^2 + Cx + D \text{ by } x - a.$$

If for convenience we write the divisor to the right of the dividend and the quotient below it, by the usual method we have:

$$\begin{array}{llll|l}
\boldsymbol{Ax^3 + Bx^2} & & \boldsymbol{+ Cx} & \boldsymbol{+ D} & \underline{x - a} \\
\underline{\boldsymbol{Ax^3 - Aax^2}} & & & & Ax^2 + (Aa + B)x \\
\quad \boldsymbol{(Aa + B)x^2} & & & & \underline{+ (Aa^2 + Ba + C)} \\
\quad (Aa + B)x^2 & \underline{\boldsymbol{-(Aa^2 + Ba)x}} & & & \\
 & \boldsymbol{(Aa^2 + Ba + C)x} & & & \\
 & (Aa^2 + Ba + C)x & \underline{\boldsymbol{-(Aa^3 + Ba^2 + Ca)}} & & \\
 & & \boldsymbol{Aa^3 + Ba^2 + Ca + D} & &
\end{array}$$

In the shorter or synthetic method, we write only the coefficients of the dividend and place a at their right, as below:

$$\begin{array}{llll}
A & B & C & D \,|\, \underline{a} \\
\underline{\quad} & \underline{\qquad Aa} & \underline{Aa^2 + Ba} & \underline{Aa^3 + Ba^2 + Ca} \\
A & Aa + B & Aa^2 + Ba + C & Aa^3 + Ba^2 + Ca + D
\end{array}$$

Multiplying A by a, writing the product under B, and adding, we obtain $Aa + B$. Multiplying this sum by a, writing the product under C, and adding, we obtain $Aa^2 + Ba + C$. In like manner the last sum is obtained.

Now A and the first two sums are respectively the coefficients of x^2, x, and x^0 in the quotient obtained above by the ordinary method, and the last sum is the remainder.

In like manner any rational integral function of x can be divided by $x - a$. If any power of x is missing, its coefficient is zero, and must be written in its place with the other coefficients.

Observe that the shorter or synthetic method of division includes only that part of the usual method given above which is in black-faced type.

Since we omit the sign $-$ before the *second* term of the divisor, we must omit also that sign before the *second* term of each product, and then *add* that term to the dividend, as in the shorter method above.

Here the *remainder* $Aa^3 + Ba^2 + Ca + D$ is the value of the dividend $Ax^3 + Bx^2 + Cx + D$ for $x = A$, which affords a second proof of § 131.

Ex. 1. Divide $2x^4 + x^3 - 29x^2 - 9x + 180$ by $x - 4$.

Write the coefficients with 4 at their right and proceed as below:

$$\begin{array}{rrrrr|l} 2 & +1 & -29 & -9 & +180 & 4 \\ & +8 & +36 & +28 & +76 & \\ \hline 2 & +9 & +7 & +19 & +256 & \end{array}$$

Hence the quotient $= 2x^3 + 9x^2 + 7x + 19$,
and the remainder, or $f(4), = 256$.

Ex. 2. Divide $2x^4 + x^3 - 29x^2 - 9x + 180$ by $x + 5$.

$$\begin{array}{rrrrr|l} 2 & +1 & -29 & -9 & +180 & -5 \\ & -10 & +45 & -80 & +445 & \\ \hline 2 & -9 & +16 & -89 & +625 & \end{array}$$

Hence the quotient $= 2x^3 - 9x^2 + 16x - 89$,
and the remainder, or $f(-5), = 625$.

Ex. 3. Divide $x^3 + 21x + 342$ by $x + 6$.

$$\begin{array}{rrrr|l} 1 & +0 & +21 & +342 & -6 \\ & -6 & +36 & -342 & \\ \hline 1 & -6 & +57 & 0 & \end{array}$$

Hence the quotient $= x^2 - 6x + 57$,
and the remainder, or $f(-6), = 0$.
Hence the division is exact, and $x + 6$ is a factor of $f(x)$.

Exercise 151.

By Horner's method

1. Divide $x^3 - 2x^2 - 4x + 8$ by $x - 3$; by $x - 2$.
2. Divide $2x^4 + 4x^3 - x^2 - 16x - 12$ by $x + 4$; by $x + 3$.

3. Divide $3x^4 - 27x^2 + 14x + 120$ by $x - 6$; by $x + 5$.

4. Find the value of $2x^4 - 3x^3 + 3x - 1$ when $x = 4$; when $x = -3$; when $x = 3$; when $x = 5$.

5. Show that one factor of $x^3 + 8x^2 + 20x + 16$ is $x + 2$, and from the quotient find the others.

6. Show that two factors of $x^4 + x^3 - 29x^2 - 9x + 180$ are $x - 3$ and $x + 3$, and find the others.

7. Show that two factors of $x^4 - 4x^3 - 8x + 32$ are $x - 2$ and $x - 4$, and find the others.

INTEGRAL RATIONAL EQUATIONS IN ONE UNKNOWN.

470. If all the terms of an integral rational equation in x are transposed to the first member and arranged in descending powers of x, we shall obtain an *equivalent* equation of the form

$$A_0x^n + A_1x^{n-1} + A_2x^{n-2} + \cdots + A_{n-1}x + A_n = 0, \quad (B)$$

where A_0, A_1, A_2, $\cdots$, A_{n-1}, A_n denote any known numbers, real, imaginary, or complex, and n denotes the degree of the equation.

Denoting the first member of (B) by $f(x)$, (B) can be written

$$f(x) = 0.$$

471. To solve equation (B), or $f(x) = 0$, by § 149 we need to factor its first member, equate each factor to zero, and solve the resulting equations. But when (B) is above the second degree in x, the first member cannot be factored *by inspection* except in certain special cases.

The methods which follow should be used when, and only when, $f(x)$ cannot be factored by inspection.

472. *If* $\boldsymbol{a}$ *is a root of the equation* $\boldsymbol{f(x) = 0}$, *that is, if* $\boldsymbol{f(a) \equiv 0}$, *then* $\boldsymbol{f(x)}$ *is divisible by* $\boldsymbol{x - a}$ (§ 131).

Conversely, *if* $\boldsymbol{f}(\boldsymbol{x})$ *is divisible by* $\boldsymbol{x}-\boldsymbol{a}$, *then* $\boldsymbol{f}(\boldsymbol{a}) \equiv \mathbf{0}$; *that is,* $\boldsymbol{a}$ *is a root of the equation* $\boldsymbol{f}(\boldsymbol{x}) = \mathbf{0}$.

E.g., if 2 is a root of the equation

$$x^3 - 2x^2 - 4x + 8 = 0, \tag{1}$$

then its first member is divisible by $x - 2$ (§ 132).

Conversely, if the first member of (1) is divisible by $x - 2$, then 2 is a root of this equation.

473. It was proved in § 148 that n linear equations in x are jointly equivalent to an equation of the nth degree in x.

In proving the converse of this theorem in the next article we assume the following theorem:

Any integral rational equation in one unknown has at least one root, real, imaginary, or complex.

Note. The proof of this theorem is too long and difficult to be given here.

474. *Any equation of the* **n***th degree in one unknown has* $\boldsymbol{n}$, *and only* $\boldsymbol{n}$, *roots.*

Proof. By § 473, the equation $f(x) = 0$ has a root.

Let a_1 denote this root; then, by § 472, $f(x)$ is divisible by $x - a_1$, so that

$$f(x) \equiv (x - a_1) f_1(x), \tag{1}$$

in which, by the laws of division, $f_1(x)$ has the form of $f(x)$, and is of the $(n-1)$th degree.

Now the equation $f_1(x) = 0$ has a root.

Denote this root by a_2; then

$$f_1(x) \equiv (x - a_2) f_2(x), \tag{2}$$

in which $f_2(x)$ is of the $(n-2)$th degree.

Repeating this process $n-1$ times, we finally obtain

$$f_{n-1}(x) = (x - a_n) A_0, \tag{n}$$

where A_0 is the coefficient of x^n in $f(x)$.

From (1), (2), $\cdots$, (n), we obtain

$$\begin{aligned} f(x) &\equiv (x-a_1)f_1(x) \\ &\equiv (x-a_1)(x-a_2)f_2(x) \\ &\cdots \qquad \cdots \qquad \cdots \\ &\equiv (x-a_1)(x-a_2)(x-a_3)\cdots(x-a_n)A_0. \end{aligned} \tag{3}$$

Hence the equation $f(x)=0$ is equivalent to the n linear equations

$$x-a_1=0,\ x-a_2=0,\ \cdots,\ x-a_n=0,$$

and therefore has n and only n roots.

From (3), it follows that any expression of the nth degree in x can be resolved into n linear factors in x.

475. Equal roots. If two or more of the factors $x-a_1$, $x-a_2$, $\cdots$, $x-a_n$ are equal, the equation $f(x)=0$ has two or more *equal* roots.

E.g., of the equation

$$(x-4)^3(x+5)^2(x-7)=0,$$

three roots are 4 each, and two are -5 each.

Ex. One root of $2x^3-5x^2-37x+60=0$ is 5. Find the others.

One root being 5, one factor of $f(x)$ is $x-5$ (§ 472).

By division the other factor is found to be $2x^2+5x-12$.

Hence the two roots required are those of the equation

$$2x^2+5x-12=0. \tag{1}$$

The roots of (1) are evidently -4 and $3/2$.

Exercise 152.

Solve each of the following equations:

1. $x^3-6x^2+10x-8=0$, one root being 4.
2. $3x^3-25x^2+42x+40=0$, one root being 5.
3. $2x^3+x^2-15x-18=0$, one root being -2.
4. $3x^3-8x^2-31x+60=0$, one root being -3.

5. $4x^3 - 9x^2 - 3x + 10 = 0$, one root being -1.

6. $x^4 + x^3 - 29x^2 - 9x + 180 = 0$, two roots being 3 and -3.

7. $x^4 - 4x^3 - 8x + 32 = 0$, two roots being 2 and 4.

8. $2x^4 - 15x^3 + 35x^2 - 30x + 8 = 0$, two roots being 1 and 2.

9. $3x^4 - 5x^3 - 17x^2 + 13x + 6 = 0$, two roots being -2 and 3.

By § 148, form the equation whose roots are:

10. The two numbers, $\pm\sqrt{-2}$.

11. The four numbers, $\pm\sqrt{-3}$, $\pm\sqrt{-5}$.

12. The four numbers, $3 \pm \sqrt{-7}$, 5, $-2/3$.

13. $3/4$, $1 \pm \sqrt{-3}$, $1 \pm \sqrt{-5}$.

14. 2, $\pm\sqrt{-1}$, $3 \pm \sqrt{-2}$.

15. 3, -4, $\sqrt{-2}$.

In each of the last six examples, observe that the coefficients of the equation obtained are all real when, and only when, the imaginary or complex roots occur in conjugate pairs. This illustrates the converse of the next article.

476. *In any integral rational equation having only* real *coefficients, imaginary or complex roots occur in conjugate pairs; that is, if* $a + bi$ *is a root, then* $a - bi$ *is also a root.*

Proof. If the coefficients in $f(x)$ are all real, then all the terms of the expression obtained by substituting $a + bi$ for x in $f(x)$ will be real except those containing odd powers of bi, which will be imaginary.

Representing the sum of all the real terms by A, and the sum of all the imaginary terms by Bi, we have

$$f(a + bi) \equiv A + Bi. \tag{1}$$

Now $f(a-bi)$ will evidently differ from $f(a+bi)$ only in the signs before the terms containing the odd powers of bi; that is, in the sign before Bi; hence

$$f(a-bi) \equiv A - Bi. \qquad (2)$$

Since $a+bi$ is a root of $f(x)=0$, from (1) we have

$$A + Bi \equiv 0.$$

Therefore $\qquad A \equiv 0$ and $B \equiv 0. \qquad$ § 279

Hence by (2), $\qquad f(a-bi) \equiv 0.$

That is, when $a+bi$ is a root of $f(x)=0$, $a-bi$ is also a root.

Ex. One root of $x^3 - 4x^2 + 4x - 3 = 0 \qquad (1)$

is $\qquad (1+\sqrt{-3})/2$; find the others.

Since $1/2 + \sqrt{-3}/2$ is a root, $1/2 - \sqrt{-3}/2$ is also a root (§ 476). Hence two factors of the first member of (1) are

$$x - 1/2 - \sqrt{-3}/2 \text{ and } x - 1/2 + \sqrt{-3}/2,$$

whose product is $(x-1/2)^2 + 3/4$, or $x^2 - x + 1$.

But $\qquad x^3 - 4x^2 + 4x - 3 \equiv (x^2 - x + 1)(x-3); \qquad (2)$

hence the third root of (1) is 3.

Identity (2) illustrates the following principle:

477. *Any rational integral function of x whose coefficients are real can be resolved into real factors, linear or quadratic in x.*

Proof. If the coefficients of $f(x)$ are real, the imaginary or complex roots of $f(x)=0$ occur in conjugate pairs, as $a+bi$ and $a-bi$; hence the complex factors of $f(x)$ occur in conjugate pairs, as $x-a-bi$ and $x-a+bi$, whose product is a *real quadratic* expression in x; that is

$$(x-a-bi)(x-a+bi) \equiv (x-a)^2 + b^2.$$

Exercise 153.

Solve each of the following equations, and find the real factors of the first member:

1. $x^3 - 6x^2 + 57x - 196 = 0$, one root being $1 - 4\sqrt{-3}$.

2. $x^3 - 6x + 9 = 0$, one root being $(3 + \sqrt{-3})/2$.

3. $x^3 - 2x^2 + 2x - 1 = 0$, one root being $(1 + \sqrt{-3})/2$.

4. $x^4 + 4x^3 + 5x^2 + 2x - 2 = 0$, one root being $-1 + i$.

5. $x^4 + 4x^3 + 6x^2 + 4x + 5 = 0$, one root being i.

6. $x^6 - x^5 + x^4 - x^2 + x - 1 = 0$, two roots being $-i$ and $(1 + \sqrt{-3})/2$.

7. Show that in an equation with commensurable real coefficients, surd roots occur in conjugate pairs; that is, if $a + \sqrt{b}$ is a root of $f(x) = 0$, $a - \sqrt{b}$ is a root also, $\sqrt{b}$ being a surd number.

All the terms in $f(a + \sqrt{b})$ will be rational except those containing odd powers of $\sqrt{b}$, which are surd. Denote the sum of all the rational terms by A and the sum of all the surd terms by $B\sqrt{b}$; then

$$f(a + \sqrt{b}) \equiv A + B\sqrt{b}.$$

Hence $f(a - \sqrt{b}) \equiv A - B\sqrt{b}$; and so on as in § 476.

8. Solve $6x^4 - 13x^3 - 35x^2 - x + 3 = 0$, one root being $2 - \sqrt{3}$.

9. Solve $x^4 - 36x^2 + 72x - 36 = 0$, one root being $3 - \sqrt{3}$.

478. The graph of $f(x)$ illustrates the fact that *equal real* roots form the connecting link between *unequal real* roots and *imaginary* or *complex* roots, and that imaginary or complex roots occur in pairs.

E.g., by slightly diminishing the term 4 of the function $x^3 - 3x^2 + 4$, its graph in fig. 4 of § 463 would be moved downward, and would then cut the axis of x in three points; by slightly increasing the term

4, the graph would be moved upward, and would then cut the axis of x in but one point.

That is, the two equal real roots of the equation

$$x^3 - 3x^2 + 4 = 0$$

would become unequal real roots or complex roots according as the known term 4 were diminished or increased.

From fig. 5 in § 463 the pupil should follow the changes in the roots of the equation

$$x^4 + x^3 - 3x^2 - x + 2 = 0,$$

(i) when the term 2 is decreased continuously to -1;
(ii) when the term 2 is increased continuously to 4.

479. An equation of the form (B) in § 470, is said to be in the **type-form** when the coefficient of x^n is 1.

E.g., $x^4 - \frac{7}{4}x^3 + \frac{3}{4}x^2 + 4 = 0$ is in the type-form.

480. *If an equation of the* **n**th *degree is in the type-form, then*

$-$ *the coefficient of* $\boldsymbol{x}^{n-1}$ = *the sum of the roots;*

the coefficient of $\boldsymbol{x}^{n-2}$ = *the sum of the products of the roots taken two at a time;*

$-$ *the coefficient of* $\boldsymbol{x}^{n-3}$ = *the sum of the products of the roots taken three at a time.*

... ...

$(-1)^n$ (*the coefficient of* $\boldsymbol{x}^0$) = *the product of the* **n** *roots.*

Proof. Let $a_1, a_2, a_3, \cdots a_n$ denote the n roots; then, by § 148, the equation can be written in the form

$$(x - a_1)(x - a_2)(x - a_3) \cdots (x - a_n) = 0. \tag{1}$$

When $n = 2$, by multiplication (1) becomes

$$x^2 - (a_1 + a_2)x + a_1a_2 = 0,$$

which proves the theorem when $n = 2$.

When $n = 3$, by multiplication (1) becomes

$$x^3 - (a_1 + a_2 + a_3)x^2 + (a_1a_2 + a_1a_3 + a_2a_3)x - a_1a_2a_3 = 0, \tag{2}$$

which proves the theorem when $n = 3$.

From the laws of multiplication it is evident that the same relation holds when $n = 4, 5, 6, \cdots$.

Observe that, if the term in x^{n-1} is wanting, the sum of the roots is 0, and if the known term is wanting, at least one root is 0.

E.g., in the equation

$$x^4 + 6x^2 - 11x - 6 = 0,$$

the sum of the roots is 0; the sum of their products taken two at a time is 6; the sum of their products taken three at a time is 11; and their product is -6.

NOTE. The coefficients in any equation are functions of the roots; and conversely, the roots are functions of the coefficients. The roots of a literal quadratic equation have been expressed in terms of the coefficients (§ 291). The roots of a literal cubic or biquadratic equation can also be expressed in terms of the coefficients, as is shown in college algebra. But the roots of a *literal* equation of the fifth or higher degree cannot be so expressed, as was proved by Abel in 1825.

Ex. Its roots being in arithmetic progression, solve

$$4x^3 - 24x^2 + 23x + 18 = 0. \qquad (1)$$

Let a denote the second term in the A. P. and b the difference; then the three roots are $a - b$, a, $a + b$. Hence their sum is $3a$; the sum of their products taken two at a time is $3a^2 - b^2$; and their product is $a(a^2 - b^2)$.

Divide (1) by 4 to reduce it to the type-form; then, by § 480, we have

$$3a = 6,\ 3a^2 - b^2 = 23/4,\ a(a^2 - b^2) = -9/2. \qquad (2)$$

Solving the first two equations in (2), we obtain $a = 2$, $b = \pm 5/2$; and these values are found to satisfy the third equation in (2).

Hence the roots are $-1/2$, 2, and $9/2$.

Exercise 154.

1. The sum of two of its roots being zero, solve

$$4x^3 + 16x^2 - 9x - 36 = 0.$$

The sum of the three roots is -4; hence the third root is -4.

2. Its roots being in arithmetic progression, solve

$$4x^3 - 12x^2 + 3x + 5 = 0.$$

3. Its roots being in geometric progression, solve

$$3x^3 - 26x^2 + 52x - 24 = 0.$$

4. One root being $1 - \sqrt{-3}$, solve

$$x^3 - 4x^2 + 8x - 8 = 0.$$

One root being $1 - \sqrt{-3}$, a second root is $1 + \sqrt{-3}$.

The sum of these two roots is 2, and the sum of the three roots is 4; hence the third root is 2.

5. By § 480, solve each of the first five examples in exercise 153.

481. If the coefficients of $f(x)$ are all $+$, $f(x) > 0$ when $x > 0$; hence, *if the coefficients of* $\boldsymbol{f(x)}$ *are all positive,* $\boldsymbol{f(x) = 0}$ *has no positive real root.*

If the coefficients of $f(x)$ are alternately $+$ and $-$; then, when $x < 0$, $f(x) > 0$ or < 0 according as n is even or odd; hence, *if the coefficients of* $\boldsymbol{f(x)}$ *are alternately* $+$ *and* $-$, $\boldsymbol{f(x) = 0}$ *has no negative real root.*

If the sum of the coefficients of $f(x)$ is zero, $f(1) = 0$; hence, *when the sum of the coefficients of* $\boldsymbol{f(x)}$ *is zero, one root of* $\boldsymbol{f(x) = 0}$ *is* $+1$.

E.g., $x^3 + 6x^2 + 11x + 6 = 0$ has no positive root, since $f(x) > 0$ when $x > 0$.

$x^3 - 6x^2 + 10x - 8 = 0$ has no negative root; since $f(x) < 0$ when $x < 0$.

$x^4 + 2x^3 - 13x^2 - 14x + 24 = 0$ has $+1$ as a root; since $f(1) = 0$.

482. *If all the coefficients of an equation in the type-form are whole numbers, any commensurable real root of the equation is an integral factor of its known term.*

E.g., any commensurable real root of the equation

$$x^3 - 6x^2 + 10x - 8 = 0$$

is an integral factor of its known term -8; that is, any such root is

$$\pm 1, \ \pm 2, \ \pm 4, \text{ or } \pm 8.$$

Proof. Let all the coefficients of the equation

$$x^n + A_1x^{n-1} + A_2x^{n-1} + \cdots + A_n = 0 \qquad (1)$$

be whole numbers, and suppose that s/t, a fractional number in its lowest terms, is one of its roots.

Substituting s/t for x, we obtain

$$\frac{s^n}{t^n} + A_1\frac{s^{n-1}}{t^{n-1}} + A_2\frac{s^{n-2}}{t^{n-2}} + \cdots + A_n \equiv 0.$$

Multiplying by t^{n-1}, and transposing, we obtain

$$s^n/t \equiv -(A_1s^{n-1} + A_2ts^{n-2} + \cdots + A_nt^{n-1}) \qquad (2)$$

Now (2) is impossible, for its first member is a fractional number in its lowest terms, and its second member is a whole number.

Hence a fractional number cannot be a root, and therefore any commensurable root must be a whole number.

Next, let a be an integral root of (1).

Substituting a for x, transposing A_n, and dividing by a, we have

$$a^{n-1} + A_1a^{n-2} + A_2a^{n-3} + \cdots + A_{n-1} \equiv -A_n/a. \qquad (3)$$

The first member of (3) is a whole number; hence the quotient A_n/a is a whole number, *i.e.*, a is an integral factor of A_n.

Ex. 1. Solve $x^3 - 6x^2 + 10x - 8 = 0$. (1)

By § 481, (1) has no negative root; hence, by § 482, any commensurable real root of (1) is $+1$, $+2$, $+4$, or $+8$, *i.e.* it is one of the *positive* integral factors of 8.

The work of determining whether $+4$ is a root can be arranged as below:

$$\begin{array}{rrrr|l} 1 & -6 & +10 & -8 & \underline{4} \\ & +4 & -8 & +8 & \\ \hline 1 & -2 & +2 & 0 & \end{array}$$

The division is exact, and the quotient is $x^2 - 2x + 2$.

Hence the roots of (1) are 4 and the roots of

$$x^2 - 2x + 2 = 0. \qquad (2)$$

Solving (2), $\qquad x = 1 \pm \sqrt{-1}.$

Hence the roots of (1) are 4 and $1 \pm \sqrt{-1}$.

Ex. 2. Solve $x^4 + 2x^3 - 13x^2 - 14x + 24 = 0.$ (1)

By § 481, one root of (1) is + 1, and by § 482 any other commensurable real root is

$$\pm 1, \ \pm 2, \ \pm 3, \ \pm 4, \ \pm 6, \ \pm 8, \ \pm 12, \text{ or } \pm 24,$$

i.e. it is one of the integral factors of 24.

$$\begin{array}{rrrrrl}
1 & +2 & -13 & -14 & +24 & \underline{|1} \\
 & +1 & +3 & -10 & -24 & \\
\hline
1 & +3 & -10 & -24 & \underline{|-2} & \\
 & -2 & -2 & +24 & & \\
\hline
1 & +1 & -12 & & &
\end{array}$$

Hence the roots of (1) are 1, -2, and the roots of

$$x^2 + x - 12 = 0.$$

Hence the roots of (1) are 1, -2, 3, and -4.

Usually it is better to try the smaller factors of A_n first.

Exercise 155.

Solve each of the following equations:

1. $x^3 + 2x^2 + 9x + 18 = 0.$
2. $x^3 - 6x^2 + 11x - 6 = 0.$
3. $x^3 - 4x^2 - 6x + 9 = 0.$
4. $x^4 - 3x^3 + x^2 + 2x = 0.$
5. $x^3 - 8x^2 + 13x - 6 = 0.$
6. $x^3 + 6x^2 + 9x + 2 = 0.$
7. $x^3 + 5x^2 - 9x - 45 = 0.$
8. $x^4 - 4x^3 - 8x + 32 = 0.$
9. $x^4 - 6x^3 + 24x - 16 = 0.$
10. $x^4 - 3x^3 - 14x^2 + 48x - 32 = 0.$
11. $x^5 - 3x^4 - 9x^3 + 21x^2 - 10x + 24 = 0.$
12. $x^3 + 2x^2 - 23x - 60 = 0.$

483. Limits of real roots. *Superior limit.* In evaluating $f(4)$ in example 1 of § 469, the sums are all positive, and they evidently would all be greater for $x > 4$.

Hence $f(x)$ can vanish only for $x < 4$; and therefore all the roots of $f(x) = 0$ are less than 4.

Hence, *if in computing the value of* $\boldsymbol{f(^+c)}$ *all the sums are positive, the real roots of* $\boldsymbol{f(x) = 0}$ *are all less than* $\boldsymbol{^+c}$.

The least integral value of ^+c which fulfils this condition is called the *superior limit* of the real roots of $f(x) = 0$.

Inferior limit. In evaluating $f(-5)$ in example 2 of § 469, the *sums* are alternately $-$ and $+$, and they evidently would all be greater arithmetically for $x < -5$. Therefore all the real roots of $f(x) = 0$ are greater than -5.

Hence, *if in computing the value of* $\boldsymbol{f(^-b)}$ *the sums are alternately* $-$ *and* $+$, *all the real roots of* $\boldsymbol{f(x) = 0}$ *are greater than* $\boldsymbol{^-b}$.

The greatest integral value of ^-b which fulfils this condition is called the *inferior limit* of the real roots of $f(x) = 0$.

Observe that the above reasoning holds when we regard a *zero sum* as either positive or negative, and that when the *last* sum is zero, the limit obtained is itself a root.

E.g., if $\quad f(x) = x^4 + 2x^3 - 13x^2 - 14x + 24 = 0; \qquad (1)$

then in evaluating $f(4)$, the sums are all $+$; and in evaluating $f(-5)$, the sums are alternately $-$ and $+$; hence the real roots of $f(x) = 0$ lie between -5 and 4.

Hence, by § 482, any commensurable roots of (1) must be

$$\pm 1,\ \pm 2,\ \pm 3, \text{ or } -4.$$

Compare this result with example 2 in § 482.

Exercise 156.

1. Show that any commensurable real root of

$$x^3 - 2x - 50 = 0$$

lies between -2 and 4; and hence is ± 1 or 2.

2. Show that any commensurable real root of

$$x^4 - 3x^2 - 75 - 10000 = 0$$

is ± 1, ± 2, ± 4, ± 5, ± 8, or 10.

3. Show that any commensurable real root of

$$x^4 - 15x^2 + 10x + 24 = 0 \tag{1}$$

is ± 1, ± 2, ± 3, or -4.

4. Find the roots of equation (1) in example 3.

Solve each of the following equations:

5. $x^4 - 9x^3 + 17x^2 + 27x - 60 = 0.$

6. $x^4 - 45x^2 - 40x + 84 = 0.$

7. $x^5 - 4x^4 - 16x^3 + 112x^2 - 208x + 128 = 0.$

8. $x^4 - x^3 - 39x^2 + 24x + 180 = 0.$

9. $x^6 + 5x^5 - 81x^4 - 85x^3 + 964x^2 + 780x - 1584 = 0.$

10. $x^7 + x^6 - 14x^5 - 14x^4 + 49x^3 + 49x^2 - 36x = 36.$

11. $x^6 - 10x^4 - 3x^2 + 108 = 0.$

12. $x^6 - 2x^5 - 7x^4 + 20x^3 - 21x^2 - 18x + 27 = 0.$

484. *To transform an equation into another whose roots shall be some multiple of those of the given one.*

Proof. If in the equation

$$x^n + A_1x^{n-1} + A_2x^{n-2} + A_3x^{n-3} + \cdots + A_n = 0, \tag{B}$$

we put $x = x_1/a$, and multiply by a^n, we obtain

$$x_1^n + A_1ax_1^{n-1} + A_2a^2x_1^{n-2} + A_3a^3x_1^{n-3} + \cdots + A_na^n = 0. \tag{2}$$

Since $x_1 = ax$, the roots of (2) are a times those of (1).

Hence, to effect the required transformation, *multiply the second term of* (B) *by the given factor, the third term by its square, and so on.*

Observe that before the rule is applied the equation must

be put in the type-form, and any missing power of x must be written with zero as its coefficient.

This theorem becomes evident also when we observe that if in equation (2) in § 480 each root is multiplied by a, the second term will be multiplied by a, the third term by a^2, and the fourth term by a^3.

The chief use of this transformation is to clear an equation of fractional coefficients.

Ex. Solve the equation

$$x^3 - \tfrac{11}{4}x^2 + \tfrac{9}{4}x - \tfrac{9}{16} = 0, \qquad (1)$$

first transforming it into another with integral coefficients.

Multiplying the second term by a, the third by a^2, the fourth by a^3, we obtain

$$x^3 - \tfrac{11}{4}ax^2 + \tfrac{9}{4}a^2x - \tfrac{9}{16}a^3 = 0. \qquad (2)$$

By inspection we discover that 4 is the least value of a which will render the coefficients of (2) integral. Putting $a = 4$, we obtain

$$x^3 - 11x^2 + 36x - 36 = 0. \qquad (3)$$

The roots of (3) are found to be 2, 3, and 6.

But the roots of (3) are four times the roots of (1); hence the roots of (1) are 1/2, 3/4, and 3/2.

Exercise 157.

Solve the following equations by transforming them into others whose commensurable real roots are whole numbers:

1. $x^3 - \frac{4}{6}x^2 - \frac{11}{36}x + \frac{5}{36} = 0.$
2. $x^3 - x^2/4 - x/2 + 1/8 = 0.$
3. $8x^3 - 26x^2 + 11x + 10 = 0.$
4. $x^3 - x^2/3 - x/36 + 1/108 = 0.$
5. $24x^3 - 52x^2 + 26x - 3 = 0.$
6. $9x^4 - 9x^3 + 5x^2 - 3x + 2/3 = 0.$
7. $x^4 - x^3/6 - x^2/12 - 13x/24 + 1/4 = 0.$
8. $2x^4 - 12x^3 + 19x^2 - 6x + 9 = 0.$

485. *If* $f(a)$ *and* $f(b)$ *are opposite in quality, an odd number of real roots of* $f(x) = 0$ *lies between* a *and* b.

If $f(a)$ *and* $f(b)$ *are like in quality, no real root, or an even number of real roots of* $f(x) = 0$ *lies between* a *and* b.

Proof. If the ordinates of two points in the graph of $f(x)$ are opposite in quality, the points are on opposite sides of the x-axis, and the part of the graph between these points must cross that axis an odd number of times (§ 466); that is, $f(x)$ is zero for an odd number of values of x between a and b.

If the ordinates of two points are like in quality, the points are on the same side of the x-axis, and the part of the graph between these points either does not cross that axis or crosses it an even number of times, touching it being regarded as crossing it twice.

E.g., in fig. 3 of § 463, the graph cuts XX' an odd number of times between A and B or A and D, and an even number of times between A and C or B and D.

In fig. 5 of § 463, the graph cuts XX' an odd number of times between A and B or B and E, and an even number of times between A and C, C and E, or A and E.

Ex. Find the first figure of each real root of the equation

$$x^3 - 4x^2 - 6x + 8 = 0. \qquad (1)$$

By §§ 474 and 476, (1) has either three or only one real root.

By Horner's method we find that:

when	$x = -2,$	$-1,$	$0,$	$1,$	$2,$	$3,$	$4,$	$5,$
	$f(x) = -4,$	$+9,$	$+8,$	$-1,$	$-12,$	$-19,$	$-16,$	$+3.$

Since $f(-2)$ and $f(-1)$ are opposite in quality, at least one root of (1) lies between -2 and -1. For like reason a second root lies between 0 and 1, and a third between 4 and 5.

Hence two roots are $-(1\cdot+)$ and $4\cdot+$, and, since $f(0.9)$ is $+$ and $f(1)$ is $-$, the third root is $0.9+$.

486. *Any equation of an odd degree in which A_0 is positive has at least one real root whose quality is opposite to that of its known term A_n.*

Proof. If $A_0 > 0$ and $f(x)$ is of an odd degree, then

$$f(-\infty) \text{ is } -,\ f(0) = A_n,\ f(+\infty) \text{ is } +.$$

Hence if A_n is positive, one root of $f(x) = 0$ lies between $-\infty$ and 0 (§ 485); and if A_n is negative, one root lies between 0 and $+\infty$.

487. *Any equation of an even degree in which A_0 is positive and the known term A_n is negative has at least one positive and one negative real root.*

Proof. If $A_0 > 0$ and $f(x)$ is of an even degree, then

$$f(-\infty) \text{ is } +,\ f(0) \text{ is } -,\ f(+\infty) \text{ is } +.$$

Hence one root of $f(x) = 0$ lies between $-\infty$ and 0, and another between 0 and $+\infty$.

Exercise 158.

Find the first figure of each real root of the equations:

1. $x^3 - 3x^2 - 4x + 11 = 0.$
2. $x^3 + x^2 - 2x - 1 = 0.$
3. $x^4 - 4x^3 - 3x + 23 = 0.$
4. $x^3 - 4x^2 - 6x = -8.$
5. $x^3 - 2x - 5 = 0.$
6. $x^3 + x - 500 = 0.$
7. $x^3 + 10x^2 + 5x = 260.$
8. $x^4 - 12x^2 + 12x - 3 = 0.$

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